HOMOLOGICAL STABILITY FOR SPACES OF SURFACES

FEDERICO CANTERO AND OSCAR RANDAL-WILLIAMS

Abstract. We study the space of oriented genus g subsurfaces of a fixed manifold M, and in particular its homological properties. We construct a "scanning map" which compares this space to the space of sections of a certain fibre bundle over M associated to its tangent bundle, and show that this map induces an isomorphism on homology in a range of degrees.

Our results are analogous to McDuff's theorem on configuration spaces, extended from 0-manifolds to 2-manifolds.

1. Introduction

Let M be a smooth manifold, not necessarily compact and possibly with boundary. Our object of study will be certain spaces of orientable surfaces in M, which we define as follows. Let Σ_g denote a closed orientable smooth surface of genus g, and $\mathrm{Emb}(\Sigma_g, M)$ denote the space of all smooth embeddings of this surface into the interior of M, equipped with the C^∞ topology. The topological group $\mathrm{Diff}^+(\Sigma_g)$ of orientation preserving diffeomorphisms acts continuously on $\mathrm{Emb}(\Sigma_g, M)$, and we define

$$\mathcal{E}(\Sigma_g, M) = \mathrm{Emb}(\Sigma_g, M)/\mathrm{Diff}^+(\Sigma_g)$$

to be the quotient space. As a set, $\mathcal{E}(\Sigma_g, M)$ is in bijection with the set of all subsets of M which are smooth manifolds diffeomorphic to Σ_g, which is why we refer to $\mathcal{E}(\Sigma_g, M)$ as the *moduli space of genus g surfaces in M*.

We will study $\mathcal{E}(\Sigma_g, M)$ using the technique called *scanning*, which compares this space of surfaces in M with a certain space of "formal surfaces in M".

Definition 1.1. For an inner product space $(V, \langle -, - \rangle)$, define

$$\mathcal{S}(V) = \mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(V)).$$

That is, we take the Grassmannian $\mathrm{Gr}_2^+(V)$ of oriented 2-planes in V, consider the tautological 2-plane bundle $\gamma_2 \subset V \times \mathrm{Gr}_2^+(V)$, and take its orthogonal complement using the inner product on V. Then we take the Thom space of this vector bundle. We will denote the point at infinity by $\emptyset \in \mathcal{S}(V)$.

If $V \to B$ is a vector bundle with metric, we let $\mathcal{S}(V) \to B$ be the fibre bundle obtained by performing this construction fibrewise to V. It has a canonical section, given by $\emptyset$ in every fibre.

We fix a Riemannian metric $\mathfrak{g}$ on M. The *space of formal surfaces* in M is defined to be

$$\Gamma_c(\mathcal{S}(TM) \to M; \emptyset),$$

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the space of sections of $\mathcal{S}(TM) \to M$ which are compactly supported, i.e. agree with the canonical section $\emptyset$ outside of a compact set and on ∂M. Such a section chooses for each point $x \in M$ a (possibly empty) oriented affine 2-dimensional subset of T_xM. The scanning construction associates to each oriented surface $\Sigma \subset M$ such a section, by—loosely speaking—assigning to each $x \in M$ the best approximation to Σ by an affine subset of T_xM.

To make this precise, we let $\mathcal{E}^\nu(\Sigma_g, M) \subset (0,\infty) \times \mathcal{E}(\Sigma_g, M)$ be the set of pairs $(\epsilon, \Sigma \subset M)$ such that the exponential map $\exp\colon \nu(\Sigma) \to M$ restricts to an embedding on each fibre of the subspace $\nu_\epsilon(\Sigma) \subset \nu(\Sigma)$ consisting of the vectors of length $< \epsilon$. We then define a map $M \times \mathcal{E}^\nu(\Sigma_g, M) \to \mathcal{S}(TM)$ by

$$(p, \epsilon, \Sigma) \longmapsto \begin{cases} \emptyset \in \mathcal{S}(T_pM) & \text{if } p \notin \exp(\nu_\epsilon(\Sigma)), \\ (D(\exp|_{T_wM}))(T_w\Sigma \perp v) \subset T_pM & \text{if } p = \exp(v) \text{ for } v \in \nu_\epsilon(\Sigma)_w, \end{cases}$$

where we consider the oriented 2-plane $T_w\Sigma$ and vector v as lying inside $T_v(T_wM)$ using the canonical isomorphism $T_v(T_wM) \cong T_wM$, and then apply the linear isomorphism $D(\exp|_{T_wM})\colon T_v(T_wM) \to T_pM$. The adjoint to this map,

$$\mathscr{S}_g\colon \mathcal{E}^\nu(\Sigma_g, M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M; \emptyset), \tag{1.1}$$

is the *scanning map*. As the forgetful map $\mathcal{E}^\nu(\Sigma_g, M) \to \mathcal{E}(\Sigma_g, M)$ is a weak homotopy equivalence, we often consider $\mathscr{S}_g$ as a map from $\mathcal{E}(\Sigma_g, M)$.

In Section 10.1 we will construct a function $\chi\colon \Gamma_c(\mathcal{S}(TM) \to M; \emptyset) \to \mathbb{Z}$ such that $\chi \circ \mathscr{S}_g$ takes constant value $2-2g$; we think of χ as sending a formal surface to its Euler characteristic, and write $\Gamma_c(\mathcal{S}(TM) \to M; \emptyset)_g = \chi^{-1}(2-2g)$. The simplest form of our theorem is then as follows.

Theorem 1.2. *If M is simply-connected and of dimension at least 5, the scanning map $\mathscr{S}_g\colon \mathcal{E}(\Sigma_g, M) \to \Gamma_c(\mathcal{S}(TM) \to M; \emptyset)_g$ induces an isomorphism on integral homology in degrees $* \leq \frac{2g-2}{3}$.*

This should be compared with the theorem of McDuff [McD75, Theorem 1.1], which can be viewed as a similar result for spaces of embeddings of 0-manifolds, i.e. configuration spaces.

Theorem 1.2 follows from two rather more technical results. Firstly, a homology stability theorem analogous to Harer stability [Har85]. To make sense of such a result it is essential to discuss surfaces with boundary, and we will shortly define spaces of surfaces with boundary inside a manifold M with non-empty boundary. A large part of the paper is devoted to proving this homology stability theorem, which requires new techniques. The second more technical result is analogous to the Madsen–Weiss theorem [MW07], and identifies the stable homology of these spaces of surfaces. To describe these results we must first give some additional definitions.

1.1. Surfaces with boundary.

Now we suppose that M has non-empty boundary ∂M, and we are given a collar $C\colon \partial M \times [0,1) \hookrightarrow M$. For $0 < \epsilon < 1$ we write C_ϵ for the restriction of C to $\partial M \times [0,\epsilon)$. Let $\Sigma_{g,b}$ be a fixed smooth oriented surface of genus g with b boundary components, and let $c\colon \partial\Sigma_{g,b} \times [0,1) \hookrightarrow \Sigma_{g,b}$ be a collar. We write c_ϵ for the restricted collar. We also fix an embedding $\delta\colon \partial\Sigma_{g,b} \hookrightarrow \partial M$, which we call a *boundary condition*.

Let $\mathrm{Emb}_\epsilon(\Sigma_{g,b}, M; \delta)$ be the set of embeddings e such that $e \circ c_\epsilon = C_\epsilon \circ (\mathrm{Id}_{[0,\epsilon)} \times \delta)$. We topologise $\mathrm{Emb}^\epsilon(\Sigma_{g,b}, M)$ with the C^∞ topology as before. Let $\mathrm{Diff}^+_\epsilon(\Sigma_{g,b})$ be

the topological group of diffeomorphisms φ of $\Sigma_{g,b}$ such that $\varphi \circ c_\epsilon = c_\epsilon$. This group acts on $\mathrm{Emb}_\epsilon(\Sigma_{g,b}, M)$, and we define

$$\mathcal{E}_\epsilon(\Sigma_{g,b}, M; \delta) = \mathrm{Emb}_\epsilon(\Sigma_{g,b}, M; \delta)/\mathrm{Diff}^+_\epsilon(\Sigma_{g,b}).$$

If $\epsilon < \epsilon'$ there is a continuous map $\mathcal{E}_{\epsilon'}(\Sigma_{g,b}, M; \delta) \hookrightarrow \mathcal{E}_\epsilon(\Sigma_{g,b}, M; \delta)$, and we write $\mathcal{E}(\Sigma_{g,b}, M; \delta)$ for the colimit taken over all ϵ.

Let $Q\colon \partial M \rightsquigarrow N$ be a cobordism which is collared at both boundaries. Then we can glue M and Q along ∂M to obtain a new manifold $M \circ Q$ (using the collars to obtain a smooth structure). Similarly, if $e\colon \Sigma_{b+b'} \hookrightarrow Q$ is an embedding of a surface with b boundary components in ∂M and b' in N (which we call the incoming and outgoing boundaries respectively) such that every component of the surface intersects the incoming boundary, and if $e(\partial_{in}\Sigma_{b+b'}) = \mathrm{Im}(\delta)$, we obtain a *gluing map*

$$\begin{aligned} \mathcal{E}(\Sigma_{g,b}, M; \delta) &\longrightarrow \mathcal{E}(\Sigma_{h,b'}, M \cup_{\partial M} Q; e|_{\partial_{\mathrm{out}}\Sigma_{b+b'}}) \\ (W \subset M) &\longmapsto (W \cup e(\Sigma_{b+b'}) \subset M \cup_{\partial M} Q) \end{aligned}$$

where the value of h depends on the combinatorics of the topology of $\Sigma_{b+b'}$ (as $\Sigma_{g,b}$ is connected and every path component of $\Sigma_{b+b'}$ intersects the incoming boundary, $\Sigma_{g,b} \cup \Sigma_{b+b'}$ is connected).

In particular, if we let $Q = \partial M \times [0,1]$ and choose a diffeomorphism $M \circ Q \cong M$ (for example by reparametrising the collar in M), we obtain gluing maps $\mathcal{E}(\Sigma_{g,b}, M; \delta) \to \mathcal{E}(\Sigma_{h,b'}, M; \delta')$.

1.2. **Homological stability.** There are three basic types of stabilisation maps which generate them all under composition. These occur when $\Sigma_{b+b'}$ is

(i) the disjoint union of a pair of pants with the legs as incoming boundary and a collection of cylinders,
(ii) the disjoint union of a pair of pants with the waist as incoming boundary and a collection of cylinders
(iii) the disjoint union of a disc with its boundary incoming and a collection of cylinders.

When these surfaces are embedded in $\partial M \times [0,1]$, we denote the corresponding gluing maps by

$$\begin{aligned} \alpha_{g,b} = \alpha_{g,b}(M; \delta, \delta')\colon \mathcal{E}(\Sigma_{g,b}, M; \delta) &\longrightarrow \mathcal{E}(\Sigma_{g+1,b-1}, M; \delta') \\ \beta_{g,b} = \beta_{g,b}(M; \delta, \delta')\colon \mathcal{E}(\Sigma_{g,b}, M; \delta) &\longrightarrow \mathcal{E}(\Sigma_{g,b+1}, M; \delta') \\ \gamma_{g,b} = \gamma_{g,b}(M; \delta, \delta')\colon \mathcal{E}(\Sigma_{g,b}, M; \delta) &\longrightarrow \mathcal{E}(\Sigma_{g,b-1}, M; \delta'). \end{aligned}$$

As a warning to the reader, we remark that *the notation does not determine the map*: we will often write, for example, $\beta_{g,b}$ to denote *any* gluing map of this type. There are many because there can be many non-isotopic embeddings of $\Sigma_{b+b'}$ into $\partial M \times [0,1]$.

The following is our main result concerning the homological stability of these spaces.

Theorem 1.3. *Let M be a simply connected manifold of dimension at least 5. If the dimension of M is 5, we assume that all the pairs of pants defining stabilisation maps are contractible in $\partial M \times [0,1]$.*

(i) *Any map* $\alpha_{g,b}$ *induces an isomorphism in homology in degrees* $* \leq \frac{2g-2}{3}$ *and an epimorphism in degrees* $* \leq \frac{2g+1}{3}$.

(ii) *Any map* $\beta_{g,b}$ *induces an isomorphism in homology in degrees* $* \leq \frac{2g-3}{3}$ *and an epimorphism in degrees* $* \leq \frac{2g}{3}$. *If one of the newly created boundaries of the pair of pants is contractible in* ∂M *then the map* $\beta_{g,b}$ *is also a monomorphism in all degrees.*

(iii) *Any map* $\gamma_{g,b}(M;\delta,\delta')$ *induces an isomorphism in homology in degrees* $* \leq \frac{2g}{3}$ *and an epimorphism in degrees* $* \leq \frac{2g+3}{3}$. *If* $b \geq 2$, *then it is always an epimorphism.*

Remark 1.4. Note that we do *not* require that ∂M is simply connected, only that M is. Thus there can be many non-isotopic boundary conditions.

These stability ranges are essentially optimal. The space $\mathcal{E}(\Sigma_{g,b},\mathbb{R}^\infty;\delta)$ is a model for the classifying space of the mapping class group of $\Gamma_{g,b}$, for which the stability ranges of Theorem 1.3 are known to be essentially optimal (cf. [Bol09]).

1.3. **Stable homology.** To identify the stable homology, we require a version of the scanning map for surfaces with boundary. We will not describe it in full detail here, but just say that it is a map

$$\mathscr{S}_{g,b}\colon \mathcal{E}(\Sigma_{g,b},M;\delta) \longrightarrow \Gamma_c(\mathcal{S}(TM)\to M;\bar{\delta})_g$$

to the space of sections of the bundle $\mathcal{S}(TM)\to M$ which are compactly supported and which in addition are equal to a fixed section $\bar{\delta}\colon \partial M \to \mathcal{S}(TM)|_{\partial M}$ on the boundary.

Theorem 1.5. *If* M *is simply connected and of dimension at least 5, the map*

$$\mathscr{S}_{g,b}\colon \mathcal{E}(\Sigma_{g,b},M;\delta) \longrightarrow \Gamma_c(\mathcal{S}(TM)\to M;\bar{\delta})_g$$

induces an isomorphism on homology in degrees $* \leq \frac{2g-2}{3}$.

In this theorem the manifold M can have empty boundary, so in particular Theorem 1.2 follows.

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2. Manifolds, spaces of surfaces, differential sets and resolutions

2.1. **Manifolds with corners.** Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, and $f : A \to B$ be a continuous map. We say that f is smooth if it extends to a smooth map from an open neighbourhood of A to $\mathbb{R}^m$. If f is a homeomorphism, we say it is a diffeomorphism if it is smooth and has smooth inverse. In particular, we have defined the notion of diffeomorphism between subsets of $\mathbb{R}^d_+ := [0,\infty)^d$.

Recall from [Cer61, Lau00] that a *manifold with corners* M of dimension d is a Hausdorff, second countable topological space locally modeled on the space $\mathbb{R}^d_+$

and its diffeomorphisms. In detail, a *smooth atlas of* M is a family of topological embeddings (called *charts*) $\{\varphi_i \colon U_i \to [0,\infty)^d\}_{i\in I}$, where each U_i is an open subset of M and if $U_i \cap U_j \neq \emptyset$, then the composite $\varphi_j\varphi_i^{-1}$ is a diffeomorphism from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$. A *smooth structure* on a Hausdorff, second countable topological space is a smooth atlas which is maximal with respect to inclusion. A k-submanifold of a manifold with corners is a subset $W \subset M$ such that for each point $p \in W$ there is a chart (U,φ) of M with $p \in U$ such that $W \cap U = \varphi^{-1}(\mathbb{R}^k_+ + v)$, where $v \in \mathbb{R}^d_+$ (see [Cer61, Definition 1.3.1]).

Let N and M be manifolds with corners. A continuous function $f \colon M \to \mathbb{R}$ is *smooth* if the composition $f \circ \varphi \colon U \to \mathbb{R}$ is smooth for any chart (U,φ) of M. A continuous map $g \colon N \to M$ is *smooth* if for each smooth function $f \colon M \to \mathbb{R}$, the composite $f \circ g$ is smooth. A *diffeomorphism* is a smooth map with smooth left and right inverse. An *embedding* is a smooth map whose image is a submanifold.

Let $p = (p_1,\dots,p_d) \in \mathbb{R}^d_+$. The *tangent space* $T_p\mathbb{R}^d_+$ is defined to be $T_p\mathbb{R}^d$. Since diffeomorphisms of $\mathbb{R}^d_+$ induce isomorphisms between tangent spaces of $\mathbb{R}^d_+$, we can define the tangent bundle of a manifold with corners following the usual procedure. A smooth map $f \colon N \to M$ induces a map $f_* \colon TN \to TM$ between tangent bundles.

If (U,φ) is a chart and $p \in U$, then the number $c(p)$ of zeros in $\varphi(p)$ is independent of the chart. The boundary of M is the subspace $\partial M = \{x \mid c(p) > 0\}$. A *connected k-face* of M is the closure of a component of the subspace $\{p \in M \mid c(p) = k\}$. A manifold with corners M is a *manifold with faces* if each $p \in M$ belongs to $c(p)$ connected 1-faces. A *face* is a (possibly empty) union of pairwise disjoint connected k-faces, for some k, and is itself a manifold with faces.

Notation 2.1. For the first seven sections, the word manifold will be used synonymously with manifold with faces.

Definition 2.2. If M is a manifold, and $\partial^0 M$ is a 1-face in M, a *collar* of $\partial^0 M$ is an embedding c of the manifold $(\partial^0 M \times [0,1), \partial^0 M \times \{0\})$ into M such that $c(x,0) = x$ and such that $c_{|(F\cap\partial^0 M)\times[0,1)}$ is a collar of $\partial^0 M \cap F$ in F for any other 1-face F. A manifold is *collared* if a 1-face $\partial^0 M$ and a collar of $\partial^0 M$ are given.

If B and M are manifolds, and c and C are collars of $\partial^0 B$ and $\partial^0 M$, then an embedding $e \colon B \to M$ is said to be *ϵ-collared* if $ec(x,t) = C(e(x),t)$ for all $t < \epsilon$. In particular $e(\partial^0 B) \subset \partial^0 M$. An embedding e is said to be *collared* if it is ϵ-collared for some $\epsilon > 0$.

A smooth map $f \colon B \to M$ between manifolds is *transverse* to a submanifold $W \subset M$ if for each $p \in B$ such that $f(p) \in W$ we have $Df(T_pB) + T_{f(p)}W = T_{f(p)}M$. A smooth map $f \colon B \to M$ to a manifold M is *transverse to ∂M* if it is transverse to any connected face. A *neat* embedding f is an embedding that is transverse to the boundary and that is collared if B is collared. A neat submanifold is the image of a neat embedding.

If A and W are submanifolds of the manifolds B and M, a *neat embedding of the pair (B,A) into the pair (W,M)* is an embedding $e \colon B \to M$ such that $e(B) \cap W = e(A)$ and such that $\dim(e(T_pB) + T_{e(p)}W) = \min\{\dim B + \dim W, \dim M\}$.

2.2. **Spaces of manifolds.** In this section we give a topology to the set of neat embeddings between two manifolds whose restriction to the boundary satisfy some conditions, as well as to the set of all compact connected oriented neat submanifolds of dimension 2 of a collared manifold M that have a prescribed behaviour near the

boundary of M. Roughly speaking, this behaviour consists on two things: first, they intersect the boundary of M in the same subset, second, they are collared along $\partial^0 M$ and share the same jet along ∂M.

We say that a pair of embeddings $f, g\colon B \to M$ have the same *incidence relation* if $f(x)$ and $g(x)$ belong to the interior of the same connected face of M for all $x \in B$. To each embedding $f\colon B \to M$ we can associate a function $[f]$ from B to the set of faces of M that sends a point x to the minimal face to which $f(x)$ belongs. Then f and g have the same incidence relation if and only if $[f] = [g]$. We denote by $\mathrm{Emb}(B, M; [f])$ the space of neat embeddings g of B in M such that $[g] = [f]$, endowed with the Whitney C^∞-topology, as in [Cer61]. If $f\colon (B, A) \to (M, W)$ is an embedding of a pair, we denote by $\mathrm{Emb}((B, A), (M, W); [f]) \subset \mathrm{Emb}(B, M; [f])$ the subspace of embeddings of the pair (B, A) into the pair (M, W) that have the same incidence relation as f.

A *boundary condition* for $\mathrm{Emb}(B, M; [f])$ is a function q that assigns to each point of B a neat submanifold of $[f](x)$ and is constant on each connected face. We denote by $\mathrm{Emb}(B, M; q) \subset \mathrm{Emb}(B, M; [f])$ the subspace of those embeddings g such that $g(x) \in q(x)$. If $A \subset B$ and $W \subset M$, then we denote by $\mathrm{Emb}((B, A), (M, W); q) \subset \mathrm{Emb}(B, M; q)$ the subspace of embeddings of the pair (B, A) into the pair (M, W).

We denote by $\mathrm{Diff}(M) \subset \mathrm{Emb}(M, M; [\mathrm{Id}])$ the space of diffeomorphisms of M that restrict to diffeomorphisms of the each connected face. If W is a submanifold, we denote by $\mathrm{Diff}(M; W) \subset \mathrm{Diff}(M)$ the subspace of those diffeomorphisms of M that restrict to a diffeomorphism of W, which is orientation-preserving if W is oriented. We denote by $\mathrm{Diff}_\partial(M)$ the space of diffeomorphisms of M that restrict to the identity on the boundary.

If $f, g\colon B \to M$ are neat embeddings, we say that f and g *have the same jet along* ∂M if $f^{-1}(\partial M) = g^{-1}(\partial M)$ and all the partial derivatives of f and g at all points in $f^{-1}(\partial M)$ agree. This defines an equivalence relation and we define the set of jets $J_\partial(B, M)$ to be the quotient of the set of all neat embeddings of B into M by the relation of having the same jet. If $d \in J_\partial(B, M)$, we denote by $\mathrm{Emb}(B, M; d)$ the space of all neat embeddings of B into M whose jet is d endowed with the Whitney C^∞-topology.

Let us denote by $T^r M$ the r-fold tangent space of M; where $T^k M := TT^{k-1}M$. Let $T^\infty M$ be the inverse limit of the projections

$$\ldots \longrightarrow T^k M \longrightarrow T^{k-1} M \longrightarrow \ldots \longrightarrow TM \longrightarrow M,$$

and write $T^\infty_\partial M := T^\infty M_{|\partial M}$. If $e \in \mathrm{Emb}(B, M; d)$ is an embedding with $e(\partial B) \subset \partial M$, then it induces a map $T^\infty e\colon T^\infty_\partial B \to T^\infty M$. By construction, the image of this map only depends on d, and this defines a map $j_B\colon J_\partial(B, M) \to \mathcal{P}(T^\infty M)$ to the power set of $T^\infty M$. We define the space of boundary conditions $\Delta(B, M)$ as the image of j_B and by $\Delta_n(M)$ the union of the images of j_B, where B runs along diffeomorphism classes of n-dimensional manifolds. If $\delta \in \Delta(B, M)$, we denote by $\mathcal{E}(B, M; \delta)$ the set of submanifolds W in M diffeomorphic to B such that $T^\infty_\partial W = \delta$. If $j_B(d) = \delta$, we endow it with a topology via the bijection

$$\mathrm{Emb}(B, M; d)/\mathrm{Diff}_\partial(B) \longrightarrow \mathcal{E}(B, M; \delta)$$

that sends a class of an embedding e to its image. If W is orientable, we write $\Delta^+(B, M)$ for the space of pairs (δ, θ) where $\delta \in \Delta(B, M)$ and θ is an orientation of δ. If $\delta \in \Delta^+(B, M)$, we write $\mathcal{E}^+(B, M; \delta)$ for the set of oriented submanifolds

W of M diffeomorphic to B such that $T_\partial^\infty W = \delta$, and we topologize it using the bijection

$$\mathrm{Emb}(B, M; d)/\mathrm{Diff}^+_\partial(B) \longrightarrow \mathcal{E}^+(B, M; \delta).$$

If $B = \Sigma$ is a collared compact connected oriented surface, this topology does not depend on the chosen d, because if $j(d) = j(d')$, then there exists a diffeomorphism h (that does not need to fix the boundary) of Σ such that the map $\mathrm{Emb}(\Sigma, M; d) \to \mathrm{Emb}(\Sigma, M; d')$ given by composing with h is a homeomorphism. Similarly, a diffeomorphism $h\colon \Sigma \to \Sigma'$ induces a homeomorphism $\mathrm{Emb}(\Sigma, M; d) \to \mathrm{Emb}(\Sigma', M; d \circ h^{-1})$, which is equivariant with respect to the induced homeomorphism $\mathrm{Diff}^+(\Sigma) \to \mathrm{Diff}^+(\Sigma')$. Therefore, this space is determined by the genus of Σ and the boundary condition δ (which describes the boundary of Σ).

Notation 2.3. If Σ has genus g and b boundary components, we use the shorter notation $\mathcal{E}^+_{g,b}(M; \delta) := \mathcal{E}^+(\Sigma, M; \delta)$ (although specifying b is redundant, as it is already given by δ). We will write $\delta^0 := \delta \cap \partial^0 M$, and we observe that δ^0 is determined by its underlying submanifold (hence, if M is a manifold with boundary i.e. $\partial M = \partial^0 M$, then $\Delta_2(M)$ is the set of compact submanifolds of dimension 1 in ∂M). We denote by $\mathrm{Diff}(M; \delta)$ the space of those diffeomorphisms that fix δ.

2.3. **Tubular neighbourhoods.** Let $V \subset M$ be a neat submanifold and denote by $\mathrm{N}_M V = TM_{|V}/TV$ the normal bundle of V in M. A *tubular neighbourhood* of V in M is a neat embedding

$$f\colon \mathrm{N}_M V \longrightarrow M$$

so that the restriction $f_{|V}$ is the inclusion $V \subset M$ and the composition

$$TV \oplus \mathrm{N}_M V \longrightarrow T(\mathrm{N}_M V)_{|V} \xrightarrow{Df} TM_{|V} \xrightarrow{\text{quot.}} \mathrm{N}_M V$$

agrees with the projection onto the second factor. If $W \subset M$ is a submanifold, and the inclusion $V \subset M$ is an embedding of pairs of $(V, V \cap W)$ into $(M, M \cap W)$, we define a *tubular neighbourhood of* V *in the pair* (M, W) as a tubular neighbourhood f of V in M such that $f \cap W$ is a tubular neighbourhood of $V \cap W$.

We may compactify $\mathrm{N}_M V$ fibrewise by adding an sphere at infinity to each fibre, obtaining the *closed normal bundle* $\overline{\mathrm{N}}_M V$ of V in M. We denote by $S(\overline{\mathrm{N}}_M V) \subset \overline{\mathrm{N}}_M V$ the subbundle of spheres at infinity. We define a *closed tubular neighbourhood* of a collared submanifold V to be an embedding of $\overline{\mathrm{N}}_M V$ into M whose restriction to $\mathrm{N}_M V$ is a tubular neighbourhood.

Note that V determines an incidence relation f for $\overline{\mathrm{N}}_M V$ in M, by assigning to each vector (x, v) the minimal face to which x belongs. We denote by $\overline{\mathrm{Tub}}(V, M) \subset \mathrm{Emb}((\overline{\mathrm{N}}_M V, V), (M, V); f)$ the subspace of tubular neighbourhoods. A boundary condition q_N for a tubular neighbourhood of V is a boundary condition for V in M, and we denote by $\overline{\mathrm{Tub}}(V, M; q_N) \subset \overline{\mathrm{Tub}}(V, M)$ the subspace of those tubular neighbourhoods t such that $t(x, v) \subset q_N(x)$. We denote by $\overline{\mathrm{Tub}}(V, (M, W); q_N) \subset \overline{\mathrm{Tub}}(V, M; q_N)$ the subspace of tubular neighbourhoods of V in the pair (M, W). Finally, if $q \subset q_N$ is a pair of boundary conditions, we denote by $\overline{\mathrm{Tub}}(V, M; (q_N, q))$ the space of tubular neighbourhoods of V in M such that the restriction to each face F is a tubular neighbourhood in the pair $(q_N(x), q(x))$, where x is any point in F.

The following lemma follows from the proof of [God07, Prop. 31], where it is stated for the space of all tubular neighbourhoods of compact submanifolds.

Lemma 2.4. *If V and W are compact submanifolds of M and q_N is a boundary condition for V in M such that $q_N(x)$ is a neighbourhood of x in the face $[V \subset M](x)$, then the spaces $\overline{\mathrm{Tub}}(V, M; q_N)$ and $\overline{\mathrm{Tub}}(V, (M, W); q_N)$ are contractible.*

Proof. Let us denote by $\mathrm{Tub}(V, M; q_N)$ the space of all non-closed, collared tubular neighbourhoods of V in M. The proof in [God07] has two steps. In the first, a tubular neighbourhood f is fixed and there is constructed a weak deformation retraction H of $\mathrm{Tub}(V, M; q_N)$ into the subspace T_f of all tubular neighbourhoods whose image is contained in $\mathrm{Im}\, f$. The second step provides a contraction of T_f to the point $\{f\}$. It is easy to see that if f is a closed, collared tubular neighbourhood of a pair, then both homotopies define homotopies for $\overline{\mathrm{Tub}}(V, (M, W); q_N)$, and the argument there applies verbatim. □

2.4. **Gluing maps between spaces of surfaces.** For a collared manifold M, we will use two kind of maps between spaces of surfaces of the form $\mathcal{E}^+_{g,b}(M; \delta)$. The first map glues a collar $\partial^0 M \times I$ to M and a surface $P \subset \partial^0 M \times I$ to the surfaces in $\mathcal{E}^+_{g,b}(M; \delta)$. For the second map, we remove a submanifold $u' \subset M$ from M. If a surface $u'' \subset u'$ is given, we may glue u'' to the surfaces in $M \setminus u'$, obtaining a map from $\mathcal{E}^+_{h,c}(M \setminus u'; \delta')$ to $\mathcal{E}^+_{g,b}(M; \delta)$, where h, c, δ' depends on the surface u''. In the following paragraphs these constructions are explained in detail.

The manifold M_1 is defined as the union of the manifold $(\partial^0 M \times [0,1])$ and the manifold M along $\partial^0 M \times \{0\}$ using the collar of M. The collar of M gives a canonical collar both to $\partial^0 M_1 := \partial^0 M \times \{1\}$ and to $\partial^0(\partial^0 M \times I) := \partial^0 M \times \{0, 1\}$. The boundary condition δ gives also boundary conditions

$$\tilde{\delta} = T^\infty_\partial(\delta^0 \times I) \in \Delta^+_2(\partial^0 M \times I)$$
$$\bar{\delta} = T^\infty_\partial(W \cup (\delta^0 \times I)) \in \Delta^+_2(M_1)$$

where W is any surface in $\mathcal{E}^+_{g,b}(M; \delta)$. Let Σ' be another collared surface with $\partial\Sigma' = \partial(\partial^0\Sigma \times I)$. For each $P \in \mathcal{E}(\Sigma', \partial^0 M \times [0,1]; \tilde{\delta})$, there is a continuous map

$$- \cup P \colon \mathcal{E}(\Sigma, M; \delta) \longrightarrow \mathcal{E}(\Sigma \cup \Sigma', M_1; \bar{\delta})$$

that sends a submanifold W to the union $W \cup P$. These are *maps of type I*.

If Σ is a compact connected oriented surface of genus g and b boundary components, then, in some cases, we will denote the map $- \cup P$ by

$$\alpha_{g,b}(M; \delta, \bar{\delta}) \colon \mathcal{E}^+_{g,b}(M; \delta) \longrightarrow \mathcal{E}^+_{g+1,b-1}(M; \bar{\delta})$$
$$\beta_{g,b}(M; \delta, \bar{\delta}) \colon \mathcal{E}^+_{g,b}(M; \delta) \longrightarrow \mathcal{E}^+_{g,b+1}(M; \bar{\delta})$$
$$\gamma_{g,b}(M; \delta, \bar{\delta}) \colon \mathcal{E}^+_{g,b}(M; \delta) \longrightarrow \mathcal{E}^+_{g,b-1}(M; \bar{\delta}).$$

depending on the genus and the number of boundary components of the surfaces in the target. Note that if Σ has no corners, then P will be a disjoint union of connected surfaces, one of them a pair of pants or a disc, and the rest diffeomorphic to cylinders. If Σ has corners, then P will be a disjoint union of strips, one of them meeting δ in two intervals, and the rest meeting δ in a single interval. Often we will write $\alpha_{g,b}(M)$, $\beta_{g,b}(M)$ and $\gamma_{g,b}(M)$ when the boundary condition is clear from the context or when we are talking about arbitrary boundary conditions.

Now we define the second type of gluing map. Let $s \subset \Sigma'$ be either empty or a closed tubular neighbourhood of an arc or a point in Σ. The complement $\Sigma \setminus s$ is again a collared manifold, but if s is an arc and M is a manifold with boundary,

its complement is no longer a manifold with boundary. This justifies working with manifolds with corners.

Let $u = (u', u'', u''')$ be a tuple given by a neat embedding $u''' \in \mathrm{Emb}(B, M; q)$, a closed tubular neighbourhood u' of u''' in M, and a (possibly empty) surface $u'' \in \mathcal{E}(s, u'; \delta[u])$ such that $\delta[u] \cap \delta = \delta[u]^0$. Then $\mathrm{cl}\,(M \setminus u')$ is a collared manifold with $\partial^0 \,\mathrm{cl}\,(M \setminus u') = \mathrm{cl}\,(\partial^0 M \setminus \partial^0 u')$. The boundary conditions δ and $\delta[u]$ give rise to a boundary condition

$$\delta(u) = T^\infty_\partial(\mathrm{cl}\,(W \setminus u'')) \in \Delta_2^+(\mathrm{cl}\, M \setminus u')$$

where $W \in \mathcal{E}^+_{g,b}(M; \delta)$ is any surface that contains u''.

The triple $u = (u', u'', u''')$ defines a map

$$\mathcal{E}(\mathrm{cl}\,\Sigma \setminus s, \mathrm{cl}\, M \setminus u'; \delta(u)) \longrightarrow \mathcal{E}(\Sigma, M; \delta)$$

that sends a submanifold W to the union $W \cup u''$. These are *maps of type II.*

Notation 2.5. First, since the map defined above is completely determined by the tuple u, we will use the notation $M(u)$ for $\mathrm{cl}\, M \setminus u'$. Second, for maps of type I, we denoted with a $\tilde{}$ the objects that we glue to the space of surface and with a $\overline{}$ the objects obtained by removing or gluing surfaces to $\mathcal{E}^+_{g,b}(M; \delta)$. For maps of type II, we denoted with brackets [ ] the objects that we remove from the space of surfaces and with parentheses ( ) the result of removing those objects. In addition, we denoted with $'''$ the submanifold, with $'$ the tubular neighbourhood and with $''$ the surface in the tubular neighbourhood. We will be consistent with these notations. Third, for maps of type I, the triple u defines triples $\tilde{u}$ and $\overline{u}$ in the manifolds $\partial^0 M \times I$ and M_1 given by $\tilde{u} = \partial^0 u \times I$ and $\overline{u} = u \cup \tilde{u}$. If we assume in addition that $P \cap \tilde{u}' = \tilde{u}''$ and that $(\partial^0 s) \times I \subset \Sigma'$ then in the diagram

$$\begin{tikzcd}
\mathcal{E}(\Sigma(s), M(u); \delta(u)) \arrow[r, dashed] \arrow[d] & \mathcal{E}((\Sigma \cup \Sigma')(\overline{s}), M_1(\overline{u}); \overline{\delta}(\overline{u})) \arrow[d] \\
\mathcal{E}(\Sigma, M; \delta) \arrow[r] & \mathcal{E}(\Sigma \cup \Sigma', M_1; \overline{\delta})
\end{tikzcd}$$

we may construct the upper horizontal arrow as $- \cup P \setminus \tilde{u}''$. As before, we will use the notation $P(\tilde{u}) = P \setminus \tilde{u}''$. We will apply these constructions three times:

In Section 3, u''' will be an embedding u''' of an interval with $\partial^0 I = \{0, 1\}$, u'' will be a strip, $\delta^0[u]$ a pair of intervals and $\tilde{u}'' \subset P$ a pair of strips.

In Section 5, we will use v's instead of u's and v''' will de an embedding of a half disc $D_+ = D^2 \cap \mathbb{R} \times \mathbb{R}_+$, with $\partial^0 D_+ = D^2 \cap \mathbb{R} \times \{0\}$, v'' will be empty, therefore $\delta^0[u]$ and $\tilde{v}''$ will also be empty.

In Section 7, we will use p's instead of u's and p''' will be an embedding of an interval with $\partial^0 I = \{0\}$, p'' will be a disc with $\delta^0[u]$ empty, hence $\tilde{p}''$ will be empty too.

2.5. **Restriction maps between spaces of manifolds.**

Definition 2.6. ([Cer61, Pal60]) Let G be a topological group. A G-space X is *G-locally retractile* if for any $x \in X$, there is a neighbourhood U of x and a continuous map (called the G-local retraction around x) $\xi \colon U \to G$ such that $y = \xi(y) \cdot x$ for all $y \in U$.

Lemma 2.7. *If X is a G-locally retractile locally connected space and $G_0 \subset G$ denotes the connected component of the identity, then X is also G_0-locally retractile.*

Proof. If $\xi\colon U \to G$ is a local retraction around $x \in X$, and $x \in U_0 \subset U$ is a connected neighbourhood of x, since $\xi(x) = \mathrm{Id}$, we deduce that $\xi_{|U}$ factors through G_0, and defines a G_0-local retraction around x. □

Lemma 2.8. *A G-equivariant map $f\colon E \to X$ onto a G-locally retractile space is a locally trivial fibration.*

Proof. For each $x \in X$ there is a neighbourhood U and a G-local retraction ξ that gives a homeomorphism

$$\begin{aligned} f^{-1}(x) \times U &\longrightarrow f^{-1}(U) \\ (z,y) &\longmapsto \xi(y)\cdot z. \end{aligned}$$ □

Lemma 2.9. *If $f\colon X \to Y$ is a G-equivariant map that has local sections and X is G-locally retractile, then Y is also locally retractile. In particular f is a locally trivial fibration.*

Proof. The composite of a local section for f and a G-local retraction for X gives a G-local retraction for Y. □

Let $P \to B$ be a principal G-bundle, let F be a left G-space. The space of compactly supported sections $\Gamma(P \to B)$ does not act on the space of compactly supported sections of the associated bundle $\Gamma(P \times_G F \to B)$. On the other hand, the space of compactly supported sections of the adjoint bundle $P \times_{\mathrm{adj}G} G \to B$ —classified by the composition $B \to G \to \mathrm{Aut}(G)$ of the classifying map of the principal bundle and the adjoint representation—, does act on $\Gamma_c(P \times_G F \to B)$.

Lemma 2.10. *If B is compact and locally compact, and F is a locally equiconnected (i.e. the inclusion of the diagonal in $F \times F$ is a cofibration) G-space that is also G-locally retractile, then $\mathrm{map}(B,F)$ is $\mathrm{map}(B,G)$-locally retractile. For a fibration $P \times_G F \to B$, the space of sections $\Gamma(E \to B)$ is $\Gamma(P \times_{\mathrm{adj}G} G \to B)$-locally retractile.*

Proof. Here is the proof of the first part, the proof of the second part follows from this one working with spaces over B.

Let $\Phi\colon G \times F \to F \times F$ be defined as $\Phi(x,g) = (x, g\cdot x)$. If we prove that there is a neighbourhood V of the diagonal $D = f(B) \times f(B) \subset F \times F$ and a global section φ of the restriction $\Phi_{|V}\colon \Phi^{-1}(V) \to V$, then we obtain a $\mathrm{map}(B,G)$-local retraction ψ around any point $f_0 \in \mathrm{map}(B,F)$ as the composition

$$A = \{f \mid (f_0 \times f)(B) \subset V\} \overset{f \mapsto f_0 \times f}{\longrightarrow} \mathrm{map}(B,V) \overset{\varphi\circ -}{\longrightarrow} \mathrm{map}(B, G \times F) \overset{\pi}{\to} \mathrm{map}(B,G)$$

where the last map is induced by the projection onto the second factor. To see that this is a local retraction, we first notice that A is an open neighbourhood of f_0 because B is compact and locally compact. Second, write $\Psi\colon \mathrm{map}(B,G) \times \{f_0\} \to \mathrm{map}(B,F)$, and

$$\begin{aligned} (\Psi\psi(f))(x) &= (\Psi\pi\varphi(f_0 \times f))(x) = \Psi\pi\varphi(f_0(x), f(x)) \\ &= (\pi\varphi(f_0(x), f(x)))\cdot f_0(x) = f(x). \end{aligned}$$

So we only need to find V and φ. Let us denote by $G_{x,y} = \{g \in G \mid gx = y\}$. Let $x_0 \in F$ and let $\xi\colon U_{x_0} \to G \times \{x_0\}$ be a local retraction around x_0. Then there is a homeomorphism $\Phi^{-1}(U_{x_0}) \to U_{x_0} \times U_{x_0} \times \Phi^{-1}(x_0,x_0) \cong U_{x_0} \times U_{x_0} \times G_{x_0,x_0}$, that sends a triple (x,y,g) to the triple $(x, y, \xi(y)^{-1}g\xi(x))$. Define V' to be the open set

$\bigcup_{x\in F} U_x \times U_x$. The restriction $\Phi_{|V'}\colon \Phi^{-1}(V') \to V'$ is locally trivial, hence a fibre bundle.

As we have assumed that F is locally equiconnected, we may take a neighbourhood deformation retract V of $D \subset F \times F$, and since $f(B)$ is compact, we may also impose that $V \subset V'$. Then in the following pullback square of fibre bundles

$$\begin{tikzcd}
\Phi^{-1}(D) \arrow[r] \arrow[d] & \Phi^{-1}(V) \arrow[d] \\
D \arrow[r] & V
\end{tikzcd}$$

the bottom map is a homotopy equivalence and the left vertical map has a global section that sends a pair (x, x) to the pair (x, e), where $e \in G$ is the identity element. Therefore the right vertical map has a global section ϕ too. $\square$

If $p\colon E \to B$ is a rank n vector bundle over a manifold B, we denote by $\mathrm{Vect}_k(E)$ the space $\Gamma(\mathrm{Gr}_k(E) \to B)$ of rank k vector subbundles of E. If $P \to B$ is the associated principal bundle, let $\mathrm{GL}(E)$ be the space of sections $\Gamma(P \times_{\mathrm{adj}(\mathrm{GL}_n(\mathbb{R}))} \mathrm{GL}_n(\mathbb{R}) \to B)$ of the adjoint bundle. If $Z \subset \partial B$ is a submanifold and $L_\partial \in \mathrm{Vect}_k(E_{|Z})$ is a vector subbundle, we denote by $\mathrm{Vect}_k(E; L_\partial) \subset \mathrm{Vect}_k(E)$ the subspace of those vector bundles whose restriction to Z is L_∂, and by $\mathrm{GL}_Z(E) \subset \mathrm{GL}(E)$ the group of bundle automorphisms of E that restrict to the identity over Z.

Lemma 2.11. *The Grassmannian* $\mathrm{Gr}_k(\mathbb{R}^n)$ *is compact, locally equiconnected and* $\mathrm{GL}_n(\mathbb{R})$*-locally retractile.*

Corollary 2.12. *If B is compact, then the space* $\mathrm{Vect}(E)$ *is* $\mathrm{GL}(E)$*-locally retractile and the space* $\mathrm{Vect}_k(E; L_\partial)$ *is* $\mathrm{GL}_Z(E)$*-locally retractile.*

The following lemma is a consequence of a more general theorem proved by Cerf [Cer61] in full generality for manifolds and, in more restricted cases, by Palais [Pal60] and Lima [Lim63] (who gave later a shorter proof).

Lemma 2.13. *(*[Cer61, 2.2.1 Théorème 5, 2.4.1 Théorème 5']*) If $f\colon B \to M$ is an embedding of a compact manifold B into a manifold M, then* $\mathrm{Emb}(B, M; [f])$ *is* $\mathrm{Diff}(M)$*-locally retractile. If d is a jet of an embedding of B into M, then* $\mathrm{Emb}(B, M; d)$ *is* $\mathrm{Diff}_\partial(M)$*-locally retractile*

Applying Lemma 2.8 to the restriction map we obtain

Corollary 2.14 ([Cer61])**.** *If $A \subset B$ is a compact submanifold and $f\colon B \to M$ is an embedding, then the restriction map*

$$\mathrm{Emb}(B, M; [f]) \longrightarrow \mathrm{Emb}(A, M; [f_{|A}]$$

is a locally trivial fibration.

We will need local retractibility for the space of surfaces in a manifold:

Lemma 2.15 ([BF81, Mic80])**.** *If B and M are manifolds, d is a jet of B in M and $\delta = j(d)$, then the quotient map* $\mathrm{Emb}(B, M; d) \to \mathrm{Emb}(B, M; d)/\mathrm{Diff}_\partial(B) := \mathcal{E}(B, M; \delta)$ *is a fibre bundle.* $\mathrm{Diff}_\partial(B)$ *may be replaced by* $\mathrm{Diff}^+_\partial(B)$ *if B is oriented.*

From Lemmas 2.9 and 2.15 we deduce that

Corollary 2.16. *The space* $\mathcal{E}^+_{g,b}(M; \delta)$ *is* $\mathrm{Diff}_\partial(M)$*-locally retractile.*

We will also need local retractibility for two more spaces.

Lemma 2.17. *Let $W \subset M$ be a submanifold, let $A \subset B$ be manifolds, and let $f\colon (B, A) \to (M, W)$ be an embedding. The space of embeddings of pairs* $\mathrm{Emb}((B, A), (M, W); [f])$ *is* $\mathrm{Diff}(M; W)$*-locally retractile.*

Proof. Let e_0 be one such embedding, and consider the diagram

$$
\begin{tikzcd}
 & \mathrm{Diff}(M;W) \times \{e_0\} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Diff}(M) \times \{e_0\} \arrow[dl] \arrow[dd] \\
\mathrm{Diff}(W) \times \{e_{0|A}\} \arrow[rr] \arrow[dd] & & \mathrm{Emb}(W, M; [W \subset M]) \arrow[dd, "\circ e_{0|A}"'] & \\
 & X \arrow[rr] \arrow[dl] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathrm{Emb}(B, M; [f]) \arrow[dl] \\
\mathrm{Emb}(A, W; [f_{|A}]) \arrow[rr] & & \mathrm{Emb}(A, M; [f_{|A}]) &
\end{tikzcd}
$$

where the space $X \subset \mathrm{Emb}(B, M; [f])$ is the subspace of those embeddings e such that $e(A) \subset W$. Hence the bottom square is a pullback square and has a natural action of $\mathrm{Diff}(M; W, [f])$. The space $\mathrm{Emb}((B, A), (M, W); [f])$ is a subspace of X and is invariant under the action of $\mathrm{Diff}(M; W)$. All the vertical maps except $\mathrm{Emb}(W, M; [W \subset M]) \to \mathrm{Emb}(A, M; [f_{|A}])$ are orbit maps. All the vertical maps except possibly $h\colon \mathrm{Diff}(M; W) \times \{e_0\} \to X$ are locally trivial fibrations by Lemmas 2.8 and 2.13. Moreover, h is the pullback of the other three vertical maps, hence is also a locally trivial fibration, so it has local sections. As the subspace $\mathrm{Emb}((B, A), (M, W); [f]) \subset X$ is $\mathrm{Diff}(M; W)$-invariant, any $\mathrm{Diff}(M; W)$-local retraction around e_0 in X gives, by restriction, a local retraction around e_0 in $\mathrm{Emb}((B, A), (M, W); [f])$. □

Corollary 2.18. *The space* $\mathrm{Emb}(B, M; q)$ *is* $\mathrm{Diff}(M; q)$*-locally retractile and the space* $\mathrm{Emb}((B, A), (M, W); q)$ *is* $\mathrm{Diff}(M; W, q)$*-locally retractile.*

Proof. Both results are obtained by applying Lemma 2.17 recursively on the submanifolds in the image of q. □

Let B and M be manifolds and let $C \subset B$ be a submanifold. Let q and q_N be boundary conditions for B in M. The set $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$ is defined as the set of triples (e, t, L), where

(i) $e \in \mathrm{Emb}(B, M; q)$ is an embedding,
(ii) $t \in \overline{\mathrm{Tub}}(e(B), M; (q_N, q))$,
(iii) $L \in \mathrm{Vect}_k(\mathrm{N}_M e(C); \mathrm{N}_{q(\partial^0 C)} e(\partial^0 C))$.

If $k = 0$, the subset C is irrelevant and we will write $\overline{\mathrm{T}}\mathrm{Emb}(B, W; q, q_N)$ for $\overline{\mathrm{T}}\mathrm{Emb}_{0,C}(B, M; q, q_N)$. Note that if $\partial^0 C \neq \emptyset$, the last condition forces $k = \dim q(\partial^0 C) - \dim \partial^0 C$.

The set $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$ has a natural action of the discretization of $\mathrm{Diff}(M; q, q_N)$ given as follows: If g is a diffeomorphism of M and t is a tubular neighbourhood as above, then g induces isomorphisms $TM_{|e(B)} \to TM_{|ge(B)}$

and $Te(B) \to Tge(B)$, hence an isomorphism $g_*\colon \mathrm{N}_M e(B) \to \mathrm{N}_M ge(B)$. We define $g(t)$ as the composite

$$\mathrm{N}_M ge(B) \xrightarrow{g_*^{-1}} \mathrm{N}_M e(B) \xrightarrow{t} M \xrightarrow{g} M.$$

We define $g(L)$ as $g_*(L)$.

Endow the set $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$ with the following topology: for a point (e_0, t_0, L_0), pick a $\mathrm{Diff}(M; q, q_N)$-local retraction $\xi\colon U_0 \to \mathrm{Diff}(M; q, q_N)$ around e_0 using Lemma 2.13 and write

$$U_{e_0} = \{(e, t, L) \in \overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N) \mid e \in U_0\}.$$

There is an injective map

$$j\colon U_{e_0} \longrightarrow \mathrm{Emb}(\overline{\mathrm{N}}_M e_0, M; q_N)$$

given by $j(e, t, L) = (\xi(e), t \circ \xi(\overline{\mathrm{N}}_M e(B)), \xi(\overline{\mathrm{N}}_M e(C))(L))$ and we give U_{e_0} the subspace topology. The U_e's cover $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$, so they define a topology. This discussion applies to the subspace $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}((B, A), (M, W); q, q_N)$ of tubular neighbourhoods of embeddings of a pair (B, A) in a pair (M, W), and this latter space has a natural action of $\mathrm{Diff}(M; W, q, q_N)$.

Lemma 2.19. *The space* $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$ *is* $\mathrm{Diff}(M; q, q_N)$*-locally retractile.*

Proof. Let $(e_0, t_0, L_0) \in \overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$. Let

$$\xi_{e_0}\colon U_{e_0} \longrightarrow \mathrm{Diff}(M; q, q_N)$$

be a local retraction around e_0 in $\mathrm{Emb}(B, M; q)$. Let

$$\xi_{t_0}\colon U_{t_0} \longrightarrow \mathrm{Diff}(M; q, q_N)$$

be a local retraction around t_0 in $\overline{\mathrm{Tub}}(e_0(B), M; (q_N, q))$. Let

$$\xi'_{L_0}\colon U_{L_0} \longrightarrow \mathrm{GL}_{\partial^0 C}(\mathrm{N}_M e(C))$$

be a local retraction around L_0 in $\mathrm{Vect}_k(\mathrm{N}_M e_0(C); \mathrm{N}_{q(\partial^0 C)} e_0(\partial^0 C))$. There is a canonical embedding $\mathrm{GL}_{\partial^0 C}(\mathrm{N}_M e_0(C)) \to \mathrm{Diff}(t_0(\overline{\mathrm{N}}_M e(C)); q, q_N)$ induced by the diffeomorphism t_0, and using a bump function we may construct a non-canonical embedding $\mathrm{Diff}(t_0(\overline{\mathrm{N}}_M e_0(C)); q, q_N) \to \mathrm{Diff}(M; q, q_N)$. We denote by

$$\xi_{L_0}\colon U_{L_0} \longrightarrow \mathrm{Diff}(M; q, q_N)$$

the composite of ξ'_{L_0} and these embeddings. Note that $\xi_{t_0}(t)(e_0) = e_0$, that $\xi_{L_0}(L)(e_0) = e_0$ and that $\xi_{L_0}(L)(t_0) = t_0$, by the definition of tubular neighbourhood and the definition of the action of $\mathrm{Diff}(M; q, q_N)$ on the space of tubular neighbourhoods. Let $U \subset \overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$ be the intersection of the inverse images of U_{e_0}, U_{t_0} and U_{L_0} under the projection maps. For a point (e, t, L) in U, let us denote $t_1 = \xi_{e_0}(e)^{-1}(t)$ and $L_1 = \xi_{t_0}(t_1)^{-1} \circ \xi_{e_0}(e)^{-1}(L)$, and define a map

$$\xi\colon U \to \mathrm{Diff}(M; q, q_N)$$

given by $\xi(e, t, L) = \xi_{e_0}(e) \circ \xi_{t_0}(t_1) \circ \xi_{L_0}(L_1)$. □

Using the proof of Lemma 2.17, and the previous lemma we obtain

Corollary 2.20. *The spaces* $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}((B, A), (M, W); q, q_N)$ *are* $\mathrm{Diff}(M; q, q_N)$*-locally retractile.*

2.6. **Resolutions.** A *semi-simplicial space*, also called Δ-*space*, is a contravariant functor

$$X_\bullet \colon \Delta_{\mathrm{inj}}^{\mathrm{op}} \longrightarrow \mathbf{Top}$$

from the category Δ_{inj} whose objects are non-empty finite ordinals and whose morphisms are injective order-preserving inclusions to the category **Top** of topological spaces. The image of the ordinal n is denoted X_n and we denote by $\partial_j \colon X_{n+1} \to X_n$ the image of the inclusion $n = \{0, 1, \dots, n-1\} \hookrightarrow \{0, 1, \dots, n\} = n+1$ that misses the element $j \in \{0, 1, \dots, n\}$. These are called *face maps* and the whole structure of $X_\bullet$ is determined by specifying the spaces X_n for each n together with the face maps in each level.

A *semi-simplicial space augmented over a topological space* X is a semi-simplicial space $X_\bullet$ together with a continuous map $\epsilon \colon X_0 \to X$ (the *augmentation* or 0-augmentation) such that $\epsilon\partial_0 = \epsilon\partial_1 \colon X_1 \to X$. Alternatively, an *augmented semi-simplicial space* is a contravariant functor

$$X_\bullet \colon \Delta_{\mathrm{inj},0}^{\mathrm{op}} \longrightarrow \mathbf{Top}$$

from the category $\Delta_{\mathrm{inj},0}$ whose objects are (possibly empty) ordinals and whose morphisms are injective order-preserving inclusions to the category **Top**. As above, we denote by $\partial_j \colon X_n \to X_{n+1}$ the image of the inclusion that misses j and we denote by ϵ the image of the unique inclusion $\emptyset \to 0$. We denote by $\epsilon_i \colon X_i \to X$ the unique composition of face maps and the augmentation map.

A *map between (augmented) semi-simplicial spaces* is a natural transformation of functors. If $X_\bullet \to X$ and $Y_\bullet \to Y$ are augmented semi-simplicial spaces, a semi-simplicial map $f_\bullet$ is equivalent to a sequence of maps $f_n : X_n \to Y_n$ such that $d_i \circ f_n = f_{n-1} \circ d_i$. We write $f : X \to Y$ for the map between augmentations.

Recall that there is a realization functor

$$|\cdot| \colon \text{Semi-simplicial spaces} \longrightarrow \mathbf{Top}$$

and we say that a semi-simplicial space $X_\bullet$ is a resolution of a topological space X if it has an augmentation to X and the map induced by the augmentation

$$|\epsilon_\bullet| \colon |X_\bullet| \longrightarrow X$$

is a weak homotopy equivalence. We say that it is a n-resolution if the induced map is n-connected (i.e. the relative homotopy groups $\pi_i(X, |X_\bullet|)$ vanish if $i \leq n$).

We will use the spectral sequences given by the skeletal filtration associated to augmented semi-simplicial spaces as they appear in [RW09]. For each augmented semi-simplicial space $\epsilon_\bullet : X_\bullet \to X$ there is a spectral sequence defined for $t \geq 0$, $s \geq -1$ whose first page is

$$E^1_{s,t} = H_t(X_s) \Longrightarrow H_{s+t+1}(X, |X_\bullet|),$$

and for each map between augmented semi-simplicial spaces $f_\bullet \colon X_\bullet \to Y_\bullet$ there is a spectral sequence defined for $t \geq 0$, $s \geq -1$ whose first page is

$$E^1_{s,t} = H_t(Y_s, X_s) \Longrightarrow H_{s+t+1}((|\epsilon^Y_\bullet|), (|\epsilon^X_\bullet|))$$

where, for a continuous map $f \colon A \to B$, we denote by M_f the mapping cylinder of f and by (f) the pair (M_f, A).

The following criteria will be widely used throughout the paper.

Criterion 2.21. [RW09, Lemma 2.1] *Let $\epsilon_\bullet : X_\bullet \to X$ be an augmented semi-simplicial space, and let $\epsilon_i \colon X_i \to X$ be the unique face maps. If each ϵ_i is a fibration and $\mathrm{Fib}_x(\epsilon_i)$ denotes its fibre at x, then the realization of the semi-simplicial space $\mathrm{Fib}_x(\epsilon_\bullet)$ is weakly homotopy equivalent to the homotopy fibre of $|\epsilon_\bullet|$ at x.*

An *augmented topological flag complex* is an augmented semi-simplicial space $\epsilon_\bullet : X_\bullet \to X$ such that

(i) the product map $X_i \to X_0 \times_X \ldots \times_X X_0$ is an open embedding,
(ii) a tuple $(x_0, \ldots, x_i)$ is in X_i if and only if for each $0 \leq j < k \leq i$ the pair $(x_j, x_k) \in X_0 \times_X X_0$ is in X_1.

Criterion 2.22. [GRW12, Theorem 6.2] *Let $\epsilon_\bullet : X_\bullet \to X$ be an augmented topological flag complex. Suppose that*

(i) *$X_0 \to X$ has local sections, that is, ϵ is surjective and for each $x_0 \in X_0$ such that $\epsilon(x_0) = x$ there is a neighbourhood U of x and a map $s \colon U \to X_0$ such that $\epsilon s(y) = y$ and $s(x) = x_0$,*
(ii) *given any finite collection $\{x_1, \ldots, x_n\} \subset X_0$ in a single fibre of ϵ over some $x \in X$, there is an x_∞ in that fibre such that each (x_i, x_∞) is a 1-simplex.*

Then $|\epsilon_\bullet| : |X_\bullet| \to X$ is a weak homotopy equivalence.

Remark 2.23. For the second condition, we could also ask that there be an x_0 such that each (x_0, x_i) is a 1-simplex, and the conclusion still holds.

Criterion 2.24. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a map of augmented semi-simplicial spaces such that $|\epsilon^X_\bullet| : |X_\bullet| \to X$ is $(l-1)$-connected and $|\epsilon^Y_\bullet| : |Y_\bullet| \to Y$ is l-connected. Suppose there is a sequence of path connected based spaces (B_i, b_i) and maps $p_i : Y_i \to B_i$, and form the map*

$$g_i : \mathrm{hofib}_{b_i}(p_i \circ f_i) \longrightarrow \mathrm{hofib}_{b_i}(p_i)$$

induced by composition with f_i. Suppose that there is a $k \leq l + 1$ such that

$$H_q(g_i) = 0 \text{ when } q + i \leq k, \text{ except if } (q, i) = (k, 0).$$

Then the map induced in homology by the composition of the inclusion of the fibre and the augmentation map

$$H_q(g_0) \longrightarrow H_q(f_0) \xrightarrow{\epsilon} H_q(f)$$

is an epimorphism in degrees $q \leq k$.

If in addition $H_k(g_0) = 0$, then $H_q(f) = 0$ in degrees $q \leq k$.

Given the data in this criterion, we will call the map of pairs $(g_i) \to (f_i)$ the *approximation over b_i*, and we will call the composition $(g_i) \to (f_i) \to (f)$ the *approximate augmentation over b_i*.

Proof. We have a homotopy fibre sequence of pairs $(g_i) \to (f_i) \to B_i$, and so a relative Serre spectral sequence

$$\widetilde{E}^2_{p,q} = H_p(B_i; \mathcal{H}_q(g_i)) \Longrightarrow H_{p+q}(f_i).$$

Since $H_q(g_i) = 0$ for all $q \leq k - i$ except $(q, i) = (k, 0)$, we have that $H_q(f_i) = 0$ for all $q + i \leq k$ except $(q, i) = (k, 0)$. Moreover, if $i = 0$, all differentials with target

or source $H_0(B;\mathcal{H}_q(g_0))$ for $q \leq k$ are trivial, and these are the only possibly non-trivial groups with total degree $p+q \leq k$, hence $H_q(g_0) \to H_0(B;\mathcal{H}_q(g_0)) \to H_q(f_0)$ is the composition of two epimorphisms if $q \leq k$.

The first page of the spectral sequence for the resolution $(f_\bullet) \to (f)$ is

$$E^1_{p,q} = H_q(f_p), \quad p \geq -1,$$

and it converges to zero in total degrees $p+q \leq l$. Since $H_q(f_p) = 0$ for all $p+q \leq k$ except $(p,q) = (0,k)$, any differential with target $E^r_{-1,q}$ for $q \leq k$ and $r \geq 2$,

$$d_r : E^r_{r-1,q-r+1} \longrightarrow E^r_{-1,q},$$

has source a quotient of $H_{q-r+1}(f_{r-1})$, which is trivial. As $k-1 \leq l$, and the spectral sequence converges to zero in total degrees $p+q \leq l$, we have that for each $q \leq k$ there is an $r \geq 1$ such that $E^r_{-1,q} = 0$, hence the homomorphisms induced by the augmentation map $d_1 \colon H_q(f_0) \to H_q(f)$ are epimorphisms in degrees $q \leq k$.

For the second part, note that in that case all epimorphisms $H_q(g_0) \to H_q(f_0) \to H_q(f)$ have trivial source when $q \leq k$, hence the target is also trivial is those degrees. □

There is one final concept which we will use rather often. We say that a pair of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a *homotopy fibre sequence* if $g \circ f$ is the constant map to a point $c \in C$, and the induced map $f_* : A \to \mathrm{hofib}_c(g)$ is a weak homotopy equivalence. For our purposes, such data can be treated as though g were a fibration and f were the inclusion of the fibre over c.

3. Resolutions of spaces of surfaces

In this section we will construct two $(g-1)$-resolutions of the space $\mathcal{E}^+_{g,b}(M;\delta)$, where M is a collared manifold with non-empty boundary and $\delta \in \Delta^+_2(M)$ is a non-empty boundary condition (in particular $b \geq 1$). We will also characterize the space of i-simplices of each resolution as the total space of a certain homotopy fibration. Afterwards we will explain how these $(g-1)$-resolutions give rise to a $(g-1)$-resolution or a g-resolution of the stabilisation maps (the connectivity of each resolution depends on the stabilisation map), and how to characterize their spaces of i-simplices.

3.1. **Resolutions of a single surface.** In the proof of [RW09, Proposition 4.1] the following semi-simplicial space was introduced: if W is a compact connected oriented surface with non-empty boundary, and k_0, k_1 are embedded intervals in ∂W, the semi-simplicial space $O(W;k_0,k_1)_\bullet$ is defined as follows: An i-simplex is a tuple $(a_0,\dots,a_i)$ of pairwise disjoint embeddings of the interval $[0,1]$ in W such that

(i) $a_j(0) \in k_0$ and $a_j(1) \in k_1$,
(ii) the complement of $a_0 \cup \dots \cup a_i$ in W is connected,
(iii) the ordering at the endpoints of the arcs is $(a_0(0),\dots,a_i(0))$ in k_0 and $(a_i(1),\dots,a_0(1))$ in k_1, where k_0 and k_1 are ordered according to the orientation of δ.

The jth face map forgets the jth arc. In order to simplify the notation we will write $[i]$ for $\{0, \dots, i\}$. The set of i-simplices is topologized as a union of components of the space $\mathrm{Emb}(I \times [i], W; q)$ with $q(x) = k_j$ if $x \in \{j\} \times [i]$ and $q(x) = M$ otherwise.

Proposition 3.1 (Proposition 4.1 of [RW09]). *The realization $|O(W; k_0, k_1)_\bullet|$ is $(g-2)$-connected, where g is the genus of W.*

3.2. **Resolutions of spaces of surfaces.** Let $\ell_0, \ell_1 \subset \partial^0 M$ be a pair of disjoint open balls that intersect δ^0 in two intervals and write $\ell = (\ell_0, \ell_1)$. There is a semi-simplicial space $\mathcal{O}_{g,b}(M; \delta, \ell)_\bullet$ (for which we write $\mathcal{O}_{g,b}(M; \delta)_\bullet$ for brevity) whose i-simplices are tuples $(W, u_0, \dots, u_i)$ with $u_j = (u_j', u_j'', u_j''')$ where

(i) $W \in \mathcal{E}^+_{g,b}(M; \delta)$,
(ii) $(u_0''', \dots, u_i''') \in O(W, \ell_0 \cap \delta, \ell_1 \cap \delta)_i$,
(iii) (u_j', u_j'') is a closed tubular neighbourhood of u_j''' in the pair (M, W), such that $u_j'(0, -) \in \ell_0$ and $u_j'(1, -) \in \ell_1$,
(iv) $u_0', \dots, u_i'$ are pairwise disjoint.

The jth face map forgets u_j, that is, it sends an i-simplex $(W, u_0, \dots, u_i)$ to the $(i-1)$-simplex $(W, u_0, \dots, \hat{u}_j, \dots, u_i)$. There is an augmentation map $\epsilon_\bullet$ to the space $\mathcal{E}^+_{g,b}(M; \delta)$ that forgets everything but W. This defines a semi-simplicial set, and we topologise the set of i-simplices as a subspace of $\mathcal{E}^+_{g,b}(M; \delta) \times \overline{\mathrm{T}}\mathrm{Emb}(I \times [i], M; q, q_N)$, where q and q_N stand for the boundary conditions

$$\begin{array}{lll} q(\{0\} \times [i]) = \ell_0 \cap \delta & q(\{1\} \times [i]) = \ell_1 \cap \delta & q(x) = M \text{ otherwise} \\ q_N(\{0\} \times [i]) = \ell_0 & q_N(\{1\} \times [i]) = \ell_1 & q(x) = M \text{ otherwise.} \end{array}$$

If we want to stress that ℓ_0 and ℓ_1 intersect the same (different) component(s) of δ, we will denote the semi-simplicial space by $\mathcal{O}^1_{g,b}(M; \delta, \ell)_\bullet$ ($\mathcal{O}^2_{g,b}(M; \delta, \ell)_\bullet$).

Proposition 3.2. *$\mathcal{O}_{g,b}(M; \delta)_\bullet$ is a $(g-1)$-resolution of $\mathcal{E}^+_{g,b}(M; \delta)$.*

Proof. In order to find the connectivity of the homotopy fibre of $\epsilon_\bullet$, we use Criterion 2.21 to assure that the semi-simplicial fibre $\mathrm{Fib}_W(\epsilon_\bullet)$ of $\epsilon_\bullet$ over a surface W is homotopy equivalent to the homotopy fibre of $|\epsilon_\bullet|$: the space $\mathcal{E}^+_{g,b}(M; \delta)$ is $\mathrm{Diff}_\partial(M)$-locally retractile by Corollary 2.16, and, as the group $\mathrm{Diff}(M; \delta, \ell)$ acts too on this space, any local retraction for $\mathrm{Diff}_\partial(M)$ gives also a local retraction for $\mathrm{Diff}(M; \delta, \ell)$. In addition, the augmentation maps ϵ_i are $\mathrm{Diff}(M; \delta, \ell)$-equivariant for all i. Therefore, by Lemma 2.8, they are locally trivial fibrations.

The i-simplices of $\mathrm{Fib}_W(\epsilon_\bullet)$ are tuples $(u_0, \dots, u_i)$ with $u_j = (u_j', u_j'', u_j''')$ where u_j''' are embeddings of an interval in W and (u_j', u_j'') are pairwise disjoint closed tubular neighbourhoods of u_j''' the pair (W, M) . Forgetting the closed tubular neighbourhoods gives a levelwise $\mathrm{Diff}(M; W, \delta, \ell)$-equivariant semi-simplicial map

$$r_\bullet \colon \mathrm{Fib}_W(\epsilon_\bullet) \longrightarrow O(W; \ell_0 \cap \delta, \ell_1 \cap \delta)_\bullet,$$

and the space of i-simplices of $O(W; \ell_0 \cap \delta, \ell_1 \cap \delta)_\bullet$ is $\mathrm{Diff}(W)$-locally retractile, and also $\mathrm{Diff}(M; W, \delta, \ell)$-locally retractile by Corollary 2.18. Therefore, by Lemma 2.8, $r_\bullet$ is a levelwise locally trivial fibration. The fibre of $r_\bullet$ over an i-simplex is a space of closed tubular neighbourhoods of arcs in the pair (M, W), which is contractible by Lemma 2.4, so $r_\bullet$ is a homotopy equivalence. As the space on the right is $(g-2)$-connected by Proposition 3.1, the result follows. $\square$

Define $A_i(M; \delta, \ell)$ to be the set of tuples $(u_0, \dots, u_i)$ with $u_j = (u_j', u_j'', u_j''')$ and

(i) the $u_j''' \colon I \to M$ are pairwise disjoint embeddings with $u_j'''(0) \in \ell_0 \cap \delta$ and $u_j'''(1) \in \ell_1 \cap \delta$, and $u_j'''(0) > u_k'''(0)$, $u_j'''(1) < u_k'''(1)$ if $j > k$.
(ii) u_j' is a closed tubular neighbourhood of u_j''' disjoint from u_k' if $j \neq k$ whose restriction to $u_j'''(\{0,1\})$ is a closed tubular neighbourhood in the pair $(\partial^0 M, \delta \cap \ell)$,
(iii) u_j'' is the restriction of u_j' to some oriented line bundle $L_j \subset \mathrm{N}_M u_j'''$ such that $L_{j|\partial u_j'''} = N_{\delta\cap\ell}(\partial u_j''')$, i.e. $L_j \in \mathrm{Gr}_1^+(\mathrm{N}_M u_j'''; \mathrm{N}_{\delta\cap\ell}(\partial u_j'''))$.

This space is in canonical bijection with a union of components of the space $\overline{\mathrm{T}}\mathrm{Emb}_{1,I\times[i]}(I \times [i], M; q, q_N)$. The bijection sends a triple (u', u'', u''') to the triple (u''', u', L_j), and we use it to give a topology to $A_i(M;\delta,\ell)$. There are restriction maps

$$\mathcal{O}_{g,b}(M;\delta,\ell)_i \longrightarrow A_i(M;\delta,\ell).$$

that send $(W, u_0, \ldots, u_i)$ to $(u_0, \ldots, u_i)$.

Proposition 3.3. *These maps are fibrations, and using the notation of Section 2.4, their fibres over a point* $\mathbf{u} = (u_0, \ldots, u_i)$ *in* $A_i(M;\delta,\ell)$ *are given by*

$$\begin{array}{rcccc} \mathcal{E}^+_{g-i-1,b+i+1}(M(\mathbf{u});\delta(\mathbf{u})) & \longrightarrow & \mathcal{O}^1_{g,b}(M;\delta,\ell)_i & \longrightarrow & A_i(M;\delta,\ell) \\ \mathcal{E}^+_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u})) & \longrightarrow & \mathcal{O}^2_{g,b}(M;\delta,\ell)_i & \longrightarrow & A_i(M;\delta,\ell). \end{array}$$

depending on how many components of δ *intersect* ℓ_0 *and* ℓ_1*.*

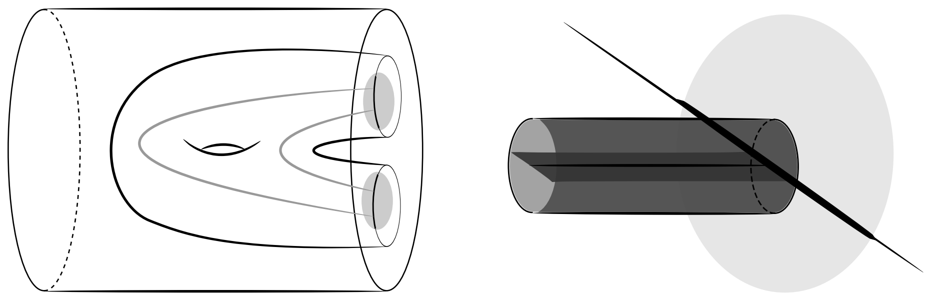
Proof. The restriction maps are $\mathrm{Diff}(M;\delta,\ell)$-equivariant, and the space $A_i(M;\delta,\ell)$ is $\mathrm{Diff}(M;\delta,\ell)$-locally retractile by Lemma 2.19, hence the map is a locally trivial fibration by Lemma 2.8.

The fibre over a point $\mathbf{u}$ is the space of surfaces W in M that contain the strips $(u_0'', \ldots, u_i'')$ and such that $W \backslash (u_0'', \ldots, u_i'')$ lies outside $u_0' \cup \ldots \cup u_i'$. If we take a parametrization $f \colon \Sigma \to W$ of any surface and write $\mathbf{a} = (a_0, \ldots, a_i) = (f^{-1} \circ u_0'', \ldots, f^{-1} \circ u_i'')$, this space is canonically homeomorphic to the space $\mathcal{E}(\Sigma(\mathbf{a}), M(\mathbf{u}); \delta(\mathbf{u}))$, so we just need to classify $\Sigma(\mathbf{a})$.

Removing a strip from Σ is the same as to removing a 1-cell, up to homotopy equivalence, hence $\chi(\Sigma(\mathbf{a})) = \chi(\Sigma) + i + 1$. Now, let us say that a strip a_j'' in Σ is of type I if both components of $\partial a_j''$ are contained in a single component of $\partial\Sigma((a_0'', \ldots, a_{j-1}''))$, and that it is of type II in other case.

- If a_j'' is of type I, then $\partial\Sigma(a_0'', \ldots, a_j'')$ has one more boundary component than δ and, as a consequence of the last condition of the definition of $O(\Sigma)_\bullet$, the strip a_{j+1} is again of type I.
- If a_j'' is of type II, then $\partial\Sigma(a_0'', \ldots, a_j'')$ has one less boundary component than δ and, as a consequence of the last condition of the definition of $O(\Sigma)_\bullet$, the strip a_{j+1}'' is of type I.

Hence, the only strip of type II that may occur in the construction of $\partial\Sigma(\mathbf{a})$ is the one given by a_0 in $O^2(\Sigma)_\bullet$. Hence $\partial\Sigma(\mathbf{a})$ has $b + i + 1$ components if $\mathbf{a} \in \overline{O}^1(\Sigma)_\bullet$ and $b + i - 1$ components if $\mathbf{a} \in \overline{O}^2(\Sigma)_\bullet$. Finally, we obtain the genus of $\Sigma(\mathbf{a})$ from the formula $g = \frac{2-\chi-b}{2}$. $\square$



(A) (B)

FIGURE 1. (A) A 2-simplex in the boundary resolution. The grey shadows are the balls ℓ_0 and ℓ_1. (b) Detail of one of the closed tubular neighbourhoods u'_j with its strip u''_j near the boundary

3.3. **Stabilisation maps between resolutions.** In this subsection we will show how to extend the stabilisation maps defined in Subsection 2.4 to maps between the resolutions we have constructed.

$$\begin{tikzcd}
\mathcal{O}^2_{g,b}(M;\delta,\ell)_i \arrow[r, dashed] \arrow[d, "\epsilon_i"] & \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_i \arrow[d, "\epsilon_i"] \\
\mathcal{E}^+_{g,b}(M,\delta) \arrow[r, "\alpha_{g,b}(M;\delta,\overline{\delta})"] & \mathcal{E}^+_{g+1,b-1}(M_1;\overline{\delta})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}^1_{g,b}(M;\delta,\ell)_i \arrow[r, dashed] \arrow[d, "\epsilon_i"] & \mathcal{O}^2_{g,b+1}(M_1;\overline{\delta},\overline{\ell})_i \arrow[d, "\epsilon_i"] \\
\mathcal{E}^+_{g,b}(M,\delta) \arrow[r, "\beta_{g,b}(M;\delta,\overline{\delta})"] & \mathcal{E}^+_{g,b+1}(M_1;\overline{\delta}).
\end{tikzcd}
\tag{3.1}$$

In Section 2.4, we defined the map $\alpha_{g,b}(M;\delta,\overline{\delta})$ by gluing a cobordism $P \subset \partial^0 M \times I$ to each surface in $\mathcal{E}^+_{g,b}(M;\delta)$. As we did there, in the following constructions we will assume, without loss of generality, that

(i) $\overline{\ell}_0 = \ell_0 \times \{1\}$ and $\overline{\ell}_1 = \ell_1 \times \{1\}$,
(ii) $P \cap (\ell \times I) = (\ell \cap \delta) \times I$, in particular $\overline{\ell} \cap \overline{\delta} = (\ell \cap \delta) \times \{1\}$.

These assumptions make the extension of the stabilisation map canonical: Let us define $\tilde{u}_j = \partial u_j \times I$. Then, joining the closed tubular neighbourhoods, strips and arcs in $(u_0, \dots, u_i)$ that live in $A_i(M;\delta,\ell)$ to the products $(\tilde{u}_0, \dots, \tilde{u}_i)$ that are subsets of $\partial^0 M \times I$, we obtain new triples $(\overline{u}_0, \dots, \overline{u}_i)$ that live in $A_i(M_1;\overline{\delta},\overline{\ell})$, where $\overline{u}_j = u \cup \tilde{u}_j$. This rule defines the dashed maps $\alpha_{g,b}(M;\delta,\overline{\delta})_i$ in the first diagram. These maps commute with the face maps and with the augmentation maps, so they define a map of semi-simplicial spaces

$$\alpha_{g,b}(M;\delta,\overline{\delta})_\bullet \colon \mathcal{O}^2_{g,b}(M;\delta,\ell)_\bullet \longrightarrow \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_\bullet.$$

which is augmented over $(\alpha_{g,b}(M;\delta,\overline{\delta}))$. Analogously, we can define a map

$$\beta_{g,b}(M;\delta,\overline{\delta})_\bullet \colon \mathcal{O}^1_{g,b}(M;\delta,\ell)_\bullet \longrightarrow \mathcal{O}^2_{g,b+1}(M_1;\overline{\delta},\overline{\ell})_\bullet$$

which is augmented over $(\beta_{g,b}(M;\delta,\overline{\delta}))$.

Corollary 3.4 (To Proposition 3.2)**.** *The semi-simplicial pair $(\alpha_{g,b}(M;\delta,\overline{\delta})_\bullet)$ together with the natural augmentation map to $(\alpha_{g,b}(M;\delta,\overline{\delta}))$ is a g-resolution. The semi-simplicial pair $(\beta_{g,b}(M;\delta,\overline{\delta})_\bullet)$ together with the natural augmentation map to $(\beta_{g,b}(M;\delta,\overline{\delta}))$ is a $(g-1)$-resolution.*

There is a commutative square

$$\begin{tikzcd}
\mathcal{O}^2_{g,b}(M;\delta,\ell)_i \arrow[r, "\alpha_{g,b}(M;\delta,\overline{\delta})_i"] \arrow[d] & \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},,\overline{\ell})_i \arrow[d] \\
A_i(M;\delta,\ell) \arrow[r, "\mathbf{u}\mapsto\overline{\mathbf{u}}"] & A_i(M_1;\overline{\delta},\overline{\ell})
\end{tikzcd} \tag{3.2}$$

where the lower map is a homotopy equivalence. Hence we obtain a map between the fibres over the points $\mathbf{u}$ and $\overline{\mathbf{u}}$ of the fibrations of Proposition 3.3,

$$\mathcal{E}^+_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u})) \longrightarrow \mathcal{E}^+_{g-i,b+i}(M_1(\overline{\mathbf{u}});\overline{\delta}(\overline{\mathbf{u}})). \tag{3.3}$$

More concretely, this is a map of type I given by the cobordism $P\backslash\tilde{\mathbf{u}}'' \subset \partial^0 M(\mathbf{u}')\times I$. Checking the difference of the genus of the surfaces in the source and target space, it follows that this map is denoted $\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}}))$.

As the map $A_i(M;\delta,\ell) \to A_i(M_1;\overline{\delta},\overline{\ell})$ is a homotopy equivalence, the space $\mathcal{E}^+_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))$ is homotopy equivalent to the homotopy fibre of the composition of the augmentation map of $\mathcal{O}^2_{g,b}(M;\delta,\ell)_\bullet$ with this map. Moreover, we have shown that the map between the fibres of the locally trivial fibrations of diagram (3.2) is a stabilisation map $\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}}))$. As a consequence we have a diagram

$$\begin{tikzcd}
\mathcal{E}^+_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u})) \arrow[rr, "\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}}))"] \arrow[d, "\simeq"] & & \mathcal{E}^+_{g-i,b+i}(M_1(\overline{\mathbf{u}});\overline{\delta}(\overline{\mathbf{u}})) \arrow[d, "\simeq"] \\
\mathrm{hofib}_{\overline{\mathbf{u}}}(\rho) \arrow[rr] \arrow[d] & & \mathrm{hofib}_{\overline{\mathbf{u}}}(\rho') \arrow[d] \\
\mathcal{O}^2_{g,b}(M;\delta,\ell)_i \arrow[rr, "\alpha_{g,b}(M;\delta,\overline{\delta})_i"] \arrow[dr, "\rho"] & & \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_i \arrow[dl, "\rho'"'] \\
& A_i(M_1;\overline{\delta},\overline{\ell}). &
\end{tikzcd}$$

This gives that the pair $(\mathrm{hofib}_{\overline{\mathbf{u}}}(\rho'),\mathrm{hofib}_{\overline{\mathbf{u}}}(\rho))$ is homotopy equivalent to the pair $(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}})))$.

Following the same procedure with the map $\beta_{g,b}(M;\delta,\overline{\delta})$, we obtain that the pair given by the map from the homotopy fibre of $\mathcal{O}^1_{g,b}(M;\delta,\ell)_i \to A_i(M_1;\overline{\delta},\overline{\ell})$ to the fibre of $\mathcal{O}^1_{g,b}(M;\delta)_i \to A_i(M_1;\overline{\delta},\overline{\ell}) \to A_i(M_1;\overline{\delta},\overline{\ell})$ is homotopy equivalent to the pair given by

$$\mathcal{E}^+_{g-i-1,b+i+1}(M(\mathbf{u});\delta(\mathbf{u})) \longrightarrow \mathcal{E}^+_{g-i,b+i}(M_1(\overline{\mathbf{u}});\overline{\delta}(\overline{\mathbf{u}})),$$

which is a map of type $\alpha_{g-i-1,b+i+1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}}))$.

Corollary 3.5 (To Proposition 3.3)**.** *There are homotopy fibre sequences*

$$\begin{aligned}
(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))) \quad &\longrightarrow \quad (\alpha_{g,b}(M;\delta)_i) \quad \longrightarrow \quad A_i(M_1;\overline{\delta},\overline{\ell}),\\
(\alpha_{g-i-1,b+i+1}(M(\mathbf{u});\delta(\mathbf{u}))) \quad &\longrightarrow \quad (\beta_{g,b}(M;\delta)_i) \quad \longrightarrow \quad A_i(M_1;\overline{\delta},\overline{\ell}),
\end{aligned}$$

that is, the homotopy fibre over $\overline{\mathbf{u}}$ is homotopy equivalent to the pair shown.

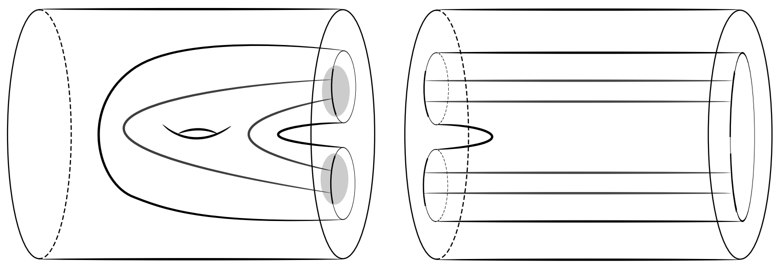


FIGURE 2. The map $\alpha_{1,2}(M)$ acting on a 2-simplex in the boundary resolution.

4. Homological stability for surfaces with boundary

In this section we will prove the first two assertions of Theorem 1.3, leaving some details to Sections 5 and 6. The proof of the last assertion will be deferred to Section 7.

Proposition 4.1. *Let M be a simply connected manifold of dimension at least 5. If the dimension of M is 5 we assume in addition that the pairs of pants defining the stabilisation maps are contractible in $\partial^0 M \times [0,1]$. Then*

(i) $H_*(\alpha_{g,b}(M)) = 0$ *for* $* \leq \frac{2g+1}{3}$,
(ii) $H_*(\beta_{g,b}(M)) = 0$ *for* $* \leq \frac{2g}{3}$.

This proposition will be proven by induction: Lemma 4.5 gives the starting step and Lemma 4.3 gives the induction step.

Remark 4.2. The proof of Proposition 4.1 follows the proof by induction of Theorem 7.1 in [RW09]. Following the language in that paper, stabilisation on π_0 is covered by Lemma 4.5 and 1-triviality will be the subject of Sections 5 and 6.

One important difference with that paper is that the fibrations in Proposition 3.3 have as fiber a space of surfaces in $M(\mathbf{u})$ —the complement of i arcs in M—, instead of M. In [RW09, Propositions 4.2 and 4.4] the fiber of the corresponding fibrations are moduli spaces of surfaces with the *same* tangential structure as the moduli space for which the $(g-1)$-resolution was constructed. This problem is solved in Section 5, where an additional resolution is introduced. This is the main technical departure from [RW09].

We will define the following statements, which we will prove by simultaneous induction: Firstly, for the stabilisation maps,

F_g $H_*(\alpha_{h,b}(M)) = 0$, for all simply connected manifolds M of dimension at least 5 with non-empty boundary, $\forall\ h \leq g$ and $\forall * \leq \frac{2h+1}{3}$,

G_g $H_*(\beta_{h,b}(M)) = 0$, for all simply connected manifolds M of dimension at least 5 with non-empty boundary, $\forall\ h \leq g$ and $\forall * \leq \frac{2h}{3}$.

Secondly, for the approximated augmentations for the g-resolution $\alpha_{g,b}(M)_\bullet$ and the $(g-1)$-resolution $\beta_{g,b}(M)_\bullet$,

- X_g $H_*(\beta_{h,b-1}(M(u))) \to H_*(\alpha_{h,b}(M))$ is an epimorphism, for all simply connected manifolds M of dimension at least 5 with non-empty boundary, $\forall h \leq g$ and $\forall * \leq \frac{2h+1}{3}$,
- A_g $H_*(\beta_{h,b-1}(M(u))) \to H_*(\alpha_{h,b}(M))$ is zero, for all simply connected manifolds M of dimension at least 5 with non-empty boundary, $\forall h \leq g$, $\forall * \leq \frac{2h+2}{3}$,
- Y_g $H_*(\alpha_{h-1,b+1}(M(u))) \to H_*(\beta_{h,b}(M))$ is an epimorphism, for all simply connected manifolds M of dimension at least 5 with non-empty boundary, $\forall h \leq g$ and $\forall * \leq \frac{2h}{3}$,
- B_g $H_*(\alpha_{h-1,b+1}(M(u))) \to H_*(\beta_{h,b}(M))$ is zero, for all simply connected manifolds M of dimension at least 5 with non-empty boundary, $\forall h \leq g$ and $\forall * \leq \frac{2h+1}{3}$.

Lemma 4.3. *If M satisfies the hypotheses of Proposition 4.1, then*

(i) $X_g, A_g \Rightarrow F_g$, (iii) $G_g \Rightarrow X_g$, (v) $G_g, X_{g-1} \Rightarrow A_g$,
(ii) $Y_g, B_g \Rightarrow G_g$, (iv) $F_{g-1} \Rightarrow Y_g$, (vi) $F_{g-1}, Y_{g-1} \Rightarrow B_g$.

Proof. (i) The morphism induced in homology by the approximate augmentation

$$H_k(\beta_{g,b-1}(M(u))) \longrightarrow H_k(\alpha_{g,b}(M))$$

is both zero and an epimorphism in all degrees $k \leq \frac{2g+1}{3}$ (since X_g and A_g hold), so $H_k(\alpha_{g,b}(M)) = 0$ in these degrees. Similarly for (ii).
(iii) Consider the g-resolution $\alpha_{g,b}(M;\delta)_\bullet$ of $\alpha_{g,b}(M;\delta)$ given by Corollary 3.4, together with the homotopy fibre sequences

$$(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))) \longrightarrow (\alpha_{g,b}(M;\delta)_i) \longrightarrow A_i(M_1;\overline{\delta})$$

of Corollary 3.5. For all $i \geq 1$ we have the inequality $\frac{2g+1}{3} - i \leq \frac{2(g-i)}{3}$, and so, as $M(\mathbf{u})$ is simply connected, by inductive assumption $H_q(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))) = 0$ for $q \leq \frac{2g+1}{3} - i$. When $i = 0$ we have the inequality $\frac{2g+1}{3} - 1 \leq \frac{2g}{3}$ so $H_q(\beta_{g,b-1}(M(\mathbf{u});\delta(\mathbf{u}))) = 0$ for $q \leq \frac{2g+1}{3} - 1$. In total we deduce that we have $H_q(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))) = 0$ for $q \leq \frac{2g+1}{3} - i$ except $(q,i) = (\frac{2g+1}{3}, 0)$. As $\lfloor \frac{2g+1}{3} \rfloor \leq g+1$, and $A_i(M_1, \overline{\delta})$ is path-connected, Criterion 2.24 shows that the approximate augmentations are epimorphisms for all $* \leq \frac{2g+1}{3}$. Similarly for (iv).
(v), (vi) These will be proven in Sections 5 and 6. □

Suppose that M is a simply connected manifold of dimension $d \geq 5$, and let us describe an action of the abelian group $H_2(M;\mathbb{Z})$ on the set $\pi_0\mathcal{E}^+_{g,b}(M;\delta)$ of isotopy classes of surfaces of genus g in M with boundary condition δ. Let $\hat{e} : \Sigma_{g,b} \hookrightarrow M$ be an embedding with boundary condition δ, representing an element $e \in \pi_0\mathcal{E}^+_{g,b}(M;\delta)$. Let $x \in \pi_2(M) \cong H_2(M;\mathbb{Z})$ be a homotopy class of maps from S^2 to M.

As M has dimension at least 5, x may be represented by an embedding $\hat{x} : S^2 \hookrightarrow M$ disjoint from the image of $\hat{e}$, and we can then choose an embedded path from the image of $\hat{e}$ to the image of $\hat{x}$. Forming the ambient connect-sum along this path we obtain a new embedding $\hat{x} \cdot \hat{e} : \Sigma_{g,b} \hookrightarrow M$.

Lemma 4.4. *The map*

$$\begin{aligned} H_2(M;\mathbb{Z}) \times \pi_0\mathcal{E}^+_{g,b}(M;\delta) &\longrightarrow \pi_0\mathcal{E}^+_{g,b}(M;\delta) \\ (x, e) &\longmapsto [\hat{x} \cdot \hat{e}] \end{aligned}$$

is well defined and gives a free and transitive action of $H_2(M;\mathbb{Z})$ *on* $\pi_0\mathcal{E}^+_{g,b}(M;\delta)$.

If $\partial\colon H_2(M,\delta;\mathbb{Z}) \to H_1(\delta;\mathbb{Z})$ *denotes the boundary homomorphism and* $[\delta] \in H_1(\delta;\mathbb{Z})$ *denotes the fundamental class, the map*

$$\begin{aligned} \pi_0\mathcal{E}^+_{g,b}(M;\delta) &\longrightarrow \partial^{-1}([\delta]) \\ [\hat{e}] &\longmapsto \hat{e}_*([\Sigma_{g,b}, \partial\Sigma_{g,b}]) \end{aligned}$$

is an isomorphism of $H_2(M;\mathbb{Z})$*-sets.*

Proof. Consider the natural $\mathrm{Diff}(\Sigma_{g,b})$-equivariant inclusion,

$$\varphi\colon \mathrm{Emb}(\Sigma_{g,b}, M;\delta) \longrightarrow \mathrm{map}(\Sigma_{g,b}, M;\delta).$$

As the dimension of M is at least 5 and it is simply connected, the main result of [Hae61] says that φ induces a bijection $\pi_0\mathrm{Emb}(\Sigma_{g,b}, M;\delta) \cong \pi_0\mathrm{map}(\Sigma_{g,b}, M;\delta)$.

Consider the cofiber sequence $S^1 \xrightarrow{i} \Sigma^1_{g,b} \to \Sigma_{g,b}$, where $\Sigma^1_{g,b}$ denotes a 1-skeleton of $\Sigma_{g,b}$ to which just a single 2-cells needs to be attached to obtain $\Sigma_{g,b}$. The second inclusion gives the locally trivial fibration

$$\mathrm{map}(\Sigma_{g,b}, M;\delta) \longrightarrow \mathrm{map}(\Sigma^1_{g,b}, M;\delta),$$

whose base space is connected because M is simply connected. The fiber over a point $\phi \in \mathrm{map}(\Sigma^1_{g,b}, M;\delta)$ is the space $\mathrm{map}(D^2, M;\phi)$ of maps from the 2-disc D^2 to M that restrict to $\phi \circ i$ on the boundary.

By considering the long exact sequence on homotopy groups for this fibration, in low degrees, we find that $\pi_1(\mathrm{map}(\Sigma^1_{g,b}, M;\delta), \phi)$ acts on the set $\pi_0(\mathrm{map}(D^2, M;\phi))$ with quotient $\pi_0(\mathrm{map}(\Sigma_{g,b}, M;\delta))$.

We have the composition

$$\pi_0(\mathrm{map}(D^2, M;\phi)) \longrightarrow\!\!\!\!\!\rightarrow \pi_0(\mathrm{map}(\Sigma_{g,b}, M;\delta)) \longrightarrow\!\!\!\!\!\rightarrow \partial^{-1}([\delta])$$

where the source has a free transitive $\pi_2(M)$-action and the target has a free transitive $H_2(M;\mathbb{Z})$-action, and the map is equivariant with respect to the Hurewicz homomorphism. This shows that both maps are in fact bijections, and that the induced $\pi_2(M)$-action on the set $\pi_0(\mathrm{map}(\Sigma_{g,b}, M;\delta))$ is free and transitive.

Given this calculation, it is clear that the group $\mathrm{Diff}^+(\Sigma_{g,b})$ acts trivially on the set $\pi_0\mathrm{Emb}(\Sigma_{g,b}, M;\delta))$, so there is an induced bijection $\pi_0\mathcal{E}^+_{g,b}(M;\delta) \to \partial^{-1}([\delta])$. It is then easy to see that the $H_2(M;\mathbb{Z})$-action on $\partial^{-1}([\delta])$ corresponds to that which we constructed on $\pi_0\mathcal{E}^+_{g,b}(M;\delta)$. □

This lemma allows us to begin the inductive proof of Proposition 4.1, as it tells us what the zeroth homology of $\mathcal{E}^+_{g,b}(M;\delta)$ is.

Lemma 4.5. *If M is simply connected of dimension at least 5, then the statements F_0 and G_0 hold. As a consequence, the statements X_0, Y_0, A_0 and B_0 hold too.*

Proof. Each stabilisation map glues on a cobordism P with incoming boundary δ and outgoing boundary $\overline{\delta}$. With the notation of Lemma 4.4, adding on the 2-chain representing the relative fundamental class of P defines an isomorphism of $H_2(M;\mathbb{Z})$-sets $\partial^{-1}([\delta]) \to \partial^{-1}([\overline{\delta}])$ between the inverse images of the fundamental classes $[\delta]$ and $[\overline{\delta}]$, and hence F_0 and G_0 hold. □

5. Resolutions of spaces of surfaces that contain an arc

In the following two sections we prove statements (v) and (vi) of Lemma 4.3. These say that the approximate augmentations over a 0-simplex u for the resolutions $(\alpha_{g,b}(M)_\bullet)$ and $(\beta_{g,b}(M)_\bullet)$, that is, the maps

$$\begin{array}{rcl}(\beta_{g,b-1}(M(u))) & \longrightarrow & (\alpha_{g,b}(M)) \\ (\alpha_{g-1,b+1}(M(u))) & \longrightarrow & (\beta_{g,b}(M)),\end{array}$$

induce the zero homomorphism in homology in degrees $* \leq \frac{2g+2}{3}$ and $\frac{2g+2}{3}$ respectively.

Let us explain the strategy of our proof of these statements, where as an example we consider the approximate augmentation for $\alpha_{g,b}(M)_\bullet$. In this section (in particular Corollary 5.5) we will construct a resolution of pairs, called the *relative disc resolution*,

$$(\mathcal{D}\beta_{g,b}(M(u))_\bullet) \quad \longrightarrow \quad (\beta_{g,b}(M(u))).$$

and in Corollary 5.6 we will show that its space of 0-simplices fits into a homotopy fibre sequence

$$(\beta_{g,b}(M(v))) \quad \longrightarrow \quad (\mathcal{D}\beta_{g,b}(M(u))_0) \quad \longrightarrow \quad D_i(M(u)),$$

where $M(v)$ denotes a manifold obtained from $M(u)$ by cutting out a 2-disc v spanning the arc u. Assuming that G_g holds, in Lemma 5.7 we will show that the *approximate augmentation for* $(\mathcal{D}\beta_{g,b}(M(u))_\bullet)$,

$$(\beta_{g,b}(M(v))) \quad \longrightarrow \quad (\mathcal{D}\beta_{g,b}(M(u))_0) \quad \longrightarrow \quad (\beta_{g,b}(M(u))),$$

induces an epimorphism in homology in degrees $* \leq \frac{2g+3}{3}$.

In Section 6 we apply the relative disc resolution as follows. Assuming X_{g-1} holds, we show that the composite of the approximate augmentation for the resolution $(\mathcal{D}\beta_{g-1,b+1}(M(u))_\bullet)$ and the approximate augmentation for the resolution $(\alpha_{g,b}(M)_\bullet)$,

$$(\beta_{g,b-1}(M(v))) \quad \longrightarrow \quad (\beta_{g,b-1}(M(u))) \quad \longrightarrow \quad (\alpha_{g,b}(M))$$

induce the zero homomorphism in homology in degrees $* \leq \frac{2g+2}{3}$. Since we have shown that the first map induces an epimorphism in homology in at least these degrees, we deduce that the approximate augmentations for the resolution $(\alpha_{g,b}(M)_\bullet)$ must induce the zero homomorphism in homology in those degrees, as required.

5.1. **Thick disc resolution.**

Suppose we have a resolution of a stabilisation map $\alpha_{g,b}(M;\delta,\overline{\delta})$ and a 0-simplex u as in Section 3.3. (The stabilisation map $\beta_{g,b}(M;\delta,\overline{\delta})$ can be treated identically). This means that we are given a pair of open balls $\ell = (\ell_0, \ell_1)$ in $\partial^0 M$ and a triple $u = (u', u'', u''') \in A_0(M;\delta,\ell)$ where u' is a tubular neighbourhood of an arc u''' in M and u'' is a strip in u' that contains u'''. In addition, the stabilisation map is a map of type I that glues a cobordism $P \subset \partial^0 M \times I$ to the surfaces in $\mathcal{E}^+_{g,b}(M;\delta)$, and P contains the strips $\tilde{u}'' = \partial u'' \times I$.

Recall that the space $\mathcal{E}^+_{g,b}(M(u);\delta(u))$, consisting of surfaces in $M(u')$ with boundary condition given by δ and the boundary of u'', was defined in Section 2.4 and it appeared in Proposition 3.3 as the fibre of the map

$$\mathcal{O}_{g,b}(M;\delta,\ell)_0 \longrightarrow A_0(M;\delta,\ell) \tag{5.1}$$

that forgets the surfaces. In Section 3.3, we showed how to extend the stabilisation map to a semi-simplicial map between resolutions, which in particular gives a map on 0-simplices

$$\mathcal{O}^2_{g,b}(M;\delta,\ell)_0 \xrightarrow{\alpha_{g,b}(M;\delta,\overline{\delta})_0} \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_0$$

This map fits also in the commutative square

$$\begin{tikzcd}
\mathcal{O}^2_{g,b}(M;\delta,\ell)_0 \arrow[r, "\alpha_{g,b}(M;\delta,\overline{\delta})_0"] \arrow[d] & \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_0 \arrow[d] \\
A_0(M;\delta,\ell) \arrow[r, "u\mapsto\overline{u}"] & A_0(M_1;\overline{\delta},\overline{\ell}),
\end{tikzcd}$$

and the maps between the fibres are

$$\beta_{g,b-1}(M;\delta,\overline{\delta})\colon \mathcal{E}^+_{g,b-1}(M(u);\delta(u)) \longrightarrow \mathcal{E}^+_{g,b}(M_1(\overline{u});\overline{\delta}(\overline{u})), \tag{5.2}$$

where $\overline{u} = u \cup \tilde{u}$ and $M_1 = M \cup \partial^0 M \times I$. This is a map of type I obtained by gluing the cobordism $P(\tilde{u}) := P \setminus \tilde{u}''$ to each surface. We will now construct a resolution of the individual spaces in (5.2), and later show how to extend the map of (5.2) to a map of resolutions.

Construction 5.1. *Let φ be a path in $P(\tilde{u})$ from $\ell_0 \cap \delta(u)$ to $\ell_1 \cap \delta(u)$ that cannot be contracted to the boundary of P. Then $\overline{\delta}$ is isotopic in $\partial^0 M$ to the oriented ambient connected sum $\#_\varphi \delta$ of δ along φ (when φ is homotoped in $\partial^0 M \times [0,1]$ to lie in $\partial^0 M \times \{0\}$).*

Let η be a path in $\partial^0 M$ from $u'''(0)$ to $u'''(1)$ that cancels the surgery done by φ, that is, so that $\#_\eta \#_\varphi \delta$ is ambient isotopic to δ.

Let $D_+ = \{(a,b) \in \mathbb{R}^2 \mid a^2+b^2 \leq 1, b \geq 0\}$ be the half disc with some collar of $\partial^0 D_+ = D_+ \cap \mathbb{R}\times\{0\}$, and write $\partial^1 D_+$ for the other 1-face of D_+. Let us fix a neat embedding (using that M is assumed to be simply connected and of dimension at least 5 [Hae61])

$$y\colon D_+ \longrightarrow M(u)$$

disjoint from δ, such that $y(\partial^1 D_+)$ is contained in $u'(S(\mathrm{N}_M u''')) \subset \partial M(u)$, and $y_{|\partial^0 D_+} \simeq \eta$ rel $(\ell_0 \cup \ell_1)$.

We remind the reader that by definition of neat embedding $y(\partial^0 D_+) \subset \partial^0 M(u)$. Choose, once for all, an (open) tubular neighbourhood Y of y disjoint from δ and write $Y^0 = Y_{|y(\partial^0 D_+)}$ and $Y^1 = Y_{|y(\partial^1 D_+)}$.

Definition 5.2. Let $\mathcal{D}_{g,b}(M(u);\delta(u),Y)_\bullet$ be the semi-simplicial space whose i-simplices are tuples $(W, v_0, \dots, v_i)$ where $v_j = (v'_j, v'''_j)$ and

- (i) $W \in \mathcal{E}^+_{g,b}(M(u);\delta(u))$,
- (ii) $v'''_j\colon D_+ \to M$ is an embedding isotopic to y with $v'''_j(\partial^0 D_+) \subset Y^0$, and $v'''_j(\partial D_+ \setminus \partial^0 D_+) \subset Y^1$,
- (iii) v'_j is a closed tubular neighbourhood of v'''_j disjoint from W, whose restriction to $v'''_j(\partial D_+)$ is contained in $Y^0 \cup Y^1$

The jth face map forgets v_j, and there is an augmentation map $\epsilon_\bullet$ to $\mathcal{E}^+_{g,b}(M(u);\delta(u))$ given by forgetting all the v_j. We topologise this set as a subspace of

$$\mathcal{E}^+_{g,b}(M(u);\delta(u)) \times \overline{\mathrm{T}}\mathrm{Emb}(D_+ \times [i], M; q, q_N),$$

where

$$q(\partial^0 D_+) = Y^0, \quad q(\partial^1 D_+) = Y^1, \quad q(x) = M \text{ otherwise}, \quad q_N = q.$$

Proposition 5.3. *If M is simply connected and of dimension at least 5, then $\mathcal{D}_{g,b}(M(u);\delta(u),Y)_\bullet$ is a resolution of $\mathcal{E}^+_{g,b}(M(u);\delta(u))$.*

Proof. In order to find the connectivity of the homotopy fibre of $\epsilon_\bullet$, we use Criterion 2.21 to assure that the semi-simplicial fibre $\mathrm{Fib}_W(\epsilon_\bullet)$ of the augmentation map $\epsilon_\bullet$ over a surface W is homotopy equivalent to the homotopy fibre of $|\epsilon_\bullet|$. The space $\mathcal{E}^+_{g,b}(M(u);\delta(u))$ is $\mathrm{Diff}_\partial(M(u))$-locally retractile by Corollary 2.16, hence also $\mathrm{Diff}(M(u);\delta(u),\partial Y)$-locally retractile. In addition, the augmentation maps ϵ_i are $\mathrm{Diff}(M;\delta(u),\partial Y)$-equivariant for all i, therefore they are also a locally trivial fibration by Lemma 2.8.

The space of i-simplices of the semi-simplicial fibre is the connected component of $\overline{\mathrm{T}}\mathrm{Emb}((D_+\times[i],\emptyset),(M,W);q,q_N)$ to which (y,Y) belongs. Let us define the semi-simplicial space $D(W,M)_\bullet$ whose space of i-simplices is $\mathrm{Emb}((D_+\times[i],\emptyset),(M,W);q)$, and the face maps are given by forgetting half-discs. Forgetting the tubular neighbourhoods gives a map

$$r_\bullet\colon \mathrm{Fib}_W(\epsilon_\bullet) \longrightarrow D(W,M)_\bullet.$$

The space on the right is *not* levelwise $\mathrm{Diff}(M;\partial Y)$-locally retractile, because an arbitrary diffeomorphism may change the isotopy class of each embedding, but is $\mathrm{Diff}_0(M;Y)$-locally retractile by Lemmas 2.7 and Corollary 2.18. In addition, $r_\bullet$ is levelwise $\mathrm{Diff}(M;\partial Y)$, hence it is a levelwise fibration by lemma 2.8. The fibre over an i-simplex $\mathbf{v}=(v_0''',\dots,v_i''')$ is the space $\overline{\mathrm{Tub}}((\mathbf{v}'''(I\times[i]),\emptyset),(M,W);q_N)$, which is contractible by Lemma 2.4.

The semi-simplicial space $D(W,M)_\bullet$ is a topological flag complex, and we will apply Criterion 2.22 to prove that it is contractible. Suppose we are given a (possibly empty) collection $v_0''',\dots,v_i'''$ of 0-simplices in $D(W,M)_\bullet$. As the dimension of M is at least 5, we may perturb the embedding y and obtain a map that is transverse to $v_0''',\dots,v_i'''$ and to W (hence disjoint). By [Hae61] and the assumption on the dimension of M, this map is homotopic to an embedding, which in turn is isotopic to y, giving a 0-simplex orthogonal to $v_0''',\dots,v_i'''$. □

Recall that $\overline{\mathrm{T}}\mathrm{Emb}(D_+\times[i],M;q,q_N)$ is the space of tuples $(v_0,\dots,v_i)$, where $v_j=(v_j',v_j''')$, with v_j''' a embedding of D_+ into M, and v_j' is a tubular neighbourhood of v_j'' as above, and such that the tubular neighbourhoods are pairwise disjoint. We denote by $D_i(M(u);Y)\subset\overline{\mathrm{T}}\mathrm{Emb}(D_+\times[i],M;q,q_N)$ the connected component of any tubular neighbourhood of the embedding y.

There is a restriction map

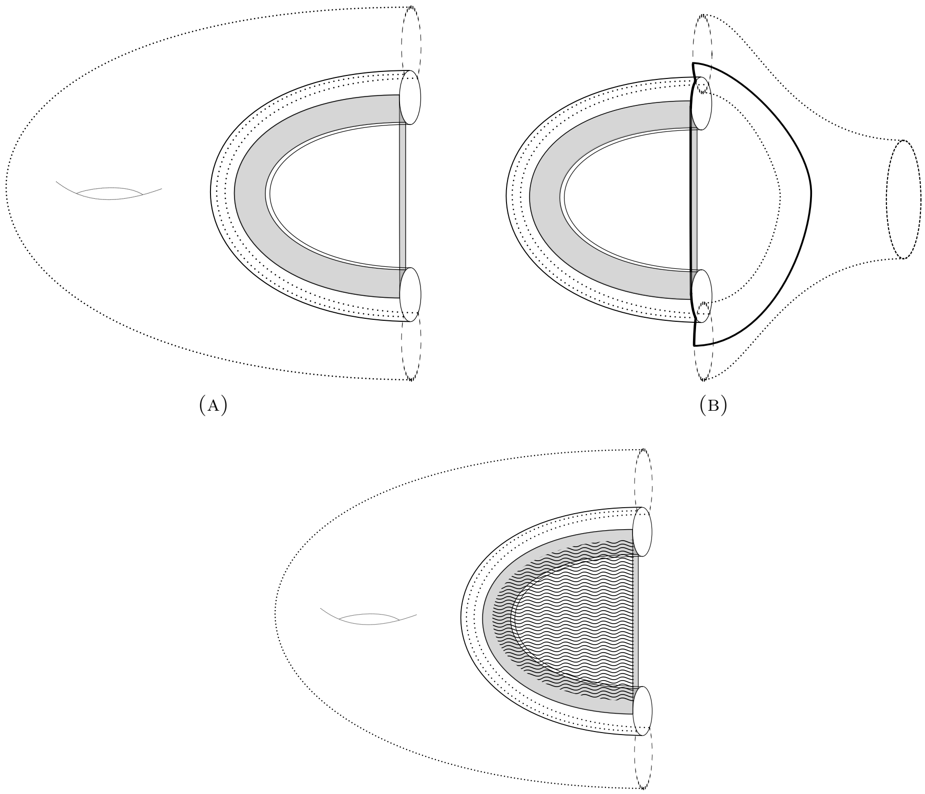
$$\mathcal{D}_{g,b}(M(u);\delta(u),Y)_i \longrightarrow D_i(M(u);Y) \tag{5.3}$$

that sends a tuple $(W,v_0,\dots,v_i)$ to the tuple $(v_0,\dots,v_i)$.

Proposition 5.4. *The map (5.3) is a locally trivial fibration, and its fibre over an i-simplex $\mathbf{v}=(v_0,\dots,v_i)$ is the space $\mathcal{E}^+_{g,b}(M(\mathbf{v});\delta(u))$ where $M(\mathbf{v}):=\mathrm{cl}\,(M(u)\backslash\mathbf{v})$.*

Proof. Since the restriction map is $\mathrm{Diff}(M(u);\delta,Y)$-equivariant, and the space $D_i(M(u);Y)$ is $\mathrm{Diff}(M(u);\delta,Y)$-locally retractile by Lemma 2.13, it follows from Lemma 2.8 that the map is a fibration.

The fibre is the space of oriented surfaces of genus g with boundary condition $\delta(u)$ in $M(u)$ that do not intersect $v'_0, \dots, v'_i$, and that space is $\mathcal{E}^+_{g,b}(M(\mathbf{v}); \delta(u))$. $\square$



(A) (B) (C)

FIGURE 3. Figure 3a shows the tubular neighbourhood u' and, in light grey, the boundary condition for the discs, that is, Y^0 and Y^1. Figure 3b corresponds to the condition that y^0 has to be homotopic to η, which in the picture can be seen as the requirement that the dark circle has to be contractible in $\partial^0 M(u)$. Figure 3c shows a typical 0-simplex in $\mathcal{D}_{1,1}(M(u); \delta(u))_\bullet$

5.2. **Stabilisation maps between resolutions.** In this section we will show how to extend the stabilisation maps of Section 3.3 to maps between the resolutions we have constructed. We will follow closely the methods of that section. As recalled in Section 5.1, to define the bottom map in the diagram

$$\begin{tikzcd}
\mathcal{D}_{g,b}(M(u); \delta(u), Y)_\bullet \arrow[r, dashed] \arrow[d] & \mathcal{D}_{g+1,b-1}(M_1(\overline{u}); \overline{\delta}(\overline{u}), \overline{Y})_\bullet \arrow[d] \\
\mathcal{E}^+_{g,b}(M(u); \delta(u)) \arrow[r, "\alpha_{g,b}(M(u);\delta(u),\overline{\delta}(\overline{u}))"] & \mathcal{E}^+_{g+1,b-1}(M_1(\overline{u}); \overline{\delta}(\overline{u}))
\end{tikzcd}$$

we joined each surface with a cobordism $P(\tilde{u})$ in $\partial^0 M(u) \times I$. We now impose, without loss of generality, the following conditions:

(i) $P(\tilde{u}) \cap (Y^0 \times I) = \emptyset$ and
(ii) $\overline{Y}^0 = Y^0 \times \{1\}$, $\overline{Y}^1 = Y^1 \cup (\partial Y^1 \times I)$.

These assumptions make the extension of the stabilisation map canonical: Let us define $\tilde{v}_j = \partial^0 v_j \times I$. Then, joining the discs in $(v_0, \dots, v_i)$ that live in $D_i(M(u); Y)$ to the products $(\tilde{v}_0, \dots, \tilde{v}_i)$ that are subsets of $\partial^0 M \times I$, we obtain new triples $(\overline{v}_0, \dots, \overline{v}_i)$ that live in $D_i(M_1; \overline{\delta})$, where $\overline{v}_j = v \cup \tilde{v}_j$. This rule defines the dashed maps $\mathcal{D}\alpha_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u}))_i$ in the first diagram.

These maps commute with the face maps and with the augmentation maps, so they define a map (the *resolution of* $\alpha_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u}))$) of semi-simplicial spaces

$$\mathcal{D}\alpha_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u}))_\bullet \colon \mathcal{D}_{g,b}(M(u); \delta(u), Y)_\bullet \longrightarrow \mathcal{D}_{g+1,b-1}(M(\overline{u}); \overline{\delta}(\overline{u}), Y)_\bullet$$

that extends $\alpha_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u}))$. Analogously, we may also define the *resolution of* $\beta_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u}))$

$$\mathcal{D}\beta_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u}))_\bullet \colon \mathcal{D}_{g,b}(M(u); \delta(u), Y)_\bullet \longrightarrow \mathcal{D}_{g,b+1}(M(\overline{u}); \overline{\delta}(\overline{u}), Y)_\bullet.$$

Corollary 5.5 (To Proposition 5.3)**.** *The pair $(\mathcal{D}\alpha_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u}))_\bullet)$ together with the natural augmentation map to the pair $(\alpha_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u})))$ is an ∞-resolution. The pair $(\mathcal{D}\beta_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u}))_\bullet)$ together with the natural augmentation map to the pair $(\beta_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u})))$ is an ∞-resolution.*

There is a commutative square,

$$\begin{tikzcd}
\mathcal{D}_{g,b}(M(u); \delta(u), Y)_i \arrow[r, "\mathcal{D}\alpha_{g,b}(M(u);\delta(u),\overline{\delta}(\overline{u}))_i"] \arrow[d] & \mathcal{D}_{g+1,b-1}(M_1(u); \overline{\delta}(\overline{u}), Y)_i \arrow[d] \\
D_i(M(u); Y) \arrow[r, "\mathbf{v} \mapsto \overline{\mathbf{v}}"] & D_i(M_1(\overline{u}); Y),
\end{tikzcd} \tag{5.4}$$

where the vertical maps are the fibrations of Proposition 5.4 and the lower map is a homotopy equivalence. Hence we obtain a map between the fibres over the points $\mathbf{v}$ and $\overline{\mathbf{v}}$ of the fibrations of (5.4)

$$\mathcal{E}^+_{g,b}(M(\mathbf{v}); \delta(u)) \longrightarrow \mathcal{E}^+_{g-1,b+1}(M_1(\overline{\mathbf{v}}); \overline{\delta}(\overline{u})),$$

which is obtained by gluing the cobordism $P(\tilde{u}) \subset \partial^0 M(\mathbf{v}) \times I$ to each surface. This is a map of type $\alpha_{g,b}(M(\mathbf{v}); \delta(u), \overline{\delta}(\overline{u}))$.

Following the same procedure with the map $\mathcal{D}\beta_{g,b}(M(u); \delta(u), \overline{\delta}(\overline{u}))_\bullet$, we obtain a map on the fibres over the points $\mathbf{v}$ and $\overline{\mathbf{v}}$ of the analogous to diagram (5.4)

$$\mathcal{E}^+_{g,b}(M(\mathbf{v}); \delta(u)) \longrightarrow \mathcal{E}^+_{g,b+1}(M_1(\overline{\mathbf{v}}); \overline{\delta}(\overline{u})),$$

which is of type $\beta_{g,b}(M(\mathbf{v}); \delta(u), \overline{\delta}(\overline{u}))$. Since the inclusion

$$D_i(M(u), Y) \longrightarrow D_i(M_1(\overline{u}), Y)$$

is a homotopy equivalence, we may compose the right hand-side fibration in diagram (5.4) with the bottom map, obtaining a map of homotopy fibrations over the same base space.

Corollary 5.6 (To Proposition 5.4)**.** *There are homotopy fibre sequences of pairs*

$$\alpha_{g,b}(M(\mathbf{v});\delta(u)) \longrightarrow \mathcal{D}\alpha_{g,b}(M(u);\delta(u))_i \longrightarrow D(M_1(\overline{u});\overline{\delta}(\overline{u}))_i$$
$$\beta_{g,b}(M(\mathbf{v});\delta(u)) \longrightarrow \mathcal{D}\beta_{g,b}(M(u);\delta(u))_i \longrightarrow D(M_1(\overline{u});\overline{\delta}(\overline{u}))_i.$$

Proof. As in Corollary 3.4, the map between the base spaces is a weak equivalence, hence we obtain a fibrewise map of fibrations. □

5.3. **Homology of approximate augmentations of the thick disc resolution.**

Lemma 5.7. *If F_g holds, then the approximate augmentations for $\mathcal{D}\alpha_{g,b}(M(u))_\bullet$ induce epimorphisms in homology up to degree $\frac{2g+4}{3}$.*

If G_g holds, then the approximate augmentations for $\mathcal{D}\beta_{g,b}(M(u))_\bullet$ induce epimorphisms in homology up to degree $\frac{2g+3}{3}$.

Proof. Consider the resolution $\mathcal{D}\alpha_{g,b}(M(u);\delta(u))_\bullet$ of $\alpha_{g,b}(M(u);\delta(u))$ given by Corollary 5.5, together with the fibrations of Corollary 5.6. The Lemma follows from Criterion 2.24 taking $k = \lfloor\frac{2g+1}{3}\rfloor + 1$ and $l = \infty$.

- For all $i \geq 1$, $\lfloor\frac{2g+1}{3}\rfloor + 1 - i \leq \frac{2g+1}{3}$ and for $i = 0$, if $q < \lfloor\frac{2g+1}{3}\rfloor + 1$ then $q \leq \frac{2g+1}{3}$, hence $H_q(\alpha_{g,b}(M(\mathbf{v});\delta(u))) = 0$ for $q \leq \lfloor\frac{2g+1}{3}\rfloor + 1 - i$ except $(q,i) = (\lfloor\frac{2g+1}{3}\rfloor + 1, 0)$.
- $\frac{2g+1}{3} \leq \infty$.

Consider the resolution $\mathcal{D}\beta_{g,b}(M(u);\delta(u))_\bullet$ of $\beta_{g,b}(M(u);\delta(u))$ given by Corollary 5.5, together with the fibrations of Corollary 5.6. The Lemma follows from Criterion 2.24 taking $k = \lfloor\frac{2g}{3}\rfloor + 1$ and $l = \infty$.

- For all $i \geq 1$, $\lfloor\frac{2g}{3}\rfloor + 1 - i \leq \frac{2g}{3}$ and for $i = 0$, if $q < \lfloor\frac{2g}{3}\rfloor + 1$ then $q \leq \frac{2g}{3}$, hence $H_q(\beta_{g,b}(M(\mathbf{v});\delta(u))) = 0$ for $q \leq \lfloor\frac{2g}{3}\rfloor + 1 - i$ except $(q,i) = (\lfloor\frac{2g}{3}\rfloor + 1, 0)$.
- $\frac{2g+1}{3} \leq \infty$. □

6. Trivial homology of approximated augmentations of the arc resolution

6.1. **The composition of the approximate augmentations.** Suppose we have a resolution of a stabilisation map $\alpha_{g,b}(M;\delta,\overline{\delta})$, a 0-simplex u in $\mathcal{O}^2_{g,b}(M;\delta,\ell)_\bullet$ and a 0-simplex v in $\mathcal{D}_{g,b}(M(u);\delta(u),Y)_\bullet$ as in Section 5.2.

Let us denote by $\mathfrak{b}_{g,b-1}(v)$ the composition of the approximate augmentation for the resolution $\mathcal{O}^2_{g,b}(M;\delta,\ell)_\bullet$ over u and the approximate augmentation for the resolution $\mathcal{D}_{g,b-1}(M(u);\delta(u),Y)_\bullet$ over v, and by $\mathfrak{a}_{g,b}(v)$ the analogue composite when we start with $\mathcal{O}^1_{g,b}(M;\delta)_\bullet$. With the notation of Sections 3.3 and 5.2, we have the following commutative square.

$$\begin{tikzcd}
\mathcal{E}^+_{g,b-1}(M(v);\delta(u)) \arrow[r, "\beta_{g,b-1}(M(v);\delta(u))"] \arrow[d, "\mathfrak{b}_{g,b-1}(v)"'] & \mathcal{E}^+_{g,b}(M_1(\overline{v});\overline{\delta}(\overline{u})) \arrow[d, "\mathfrak{a}_{g,b}(v)"] \arrow[dl, dashed] \\
\mathcal{E}^+_{g,b}(M;\delta) \arrow[r, "\alpha_{g,b}(M;\delta)"] & \mathcal{E}^+_{g+1,b-1}(M_1;\overline{\delta})
\end{tikzcd} \tag{6.1}$$

The first result of this section will be the construction of a dotted map with certain properties making both triangles commute up to homotopy. We will construct

it in Lemma 6.1. All maps in the diagram are maps between spaces of surfaces of the two kind of maps constructed in Section 2.2:

$$\begin{array}{rclllll}
\alpha_{g,b}(M;\delta) & = & -\cup P, & \text{with} & P & \subset & \partial^0 M\times I & (\text{type } I),\\
\beta_{g,b-1}(M(v);\delta(u)) & = & -\cup P(\tilde u''), & \text{with} & P(\tilde u) & \subset & \partial^0 M(v)\times I & (\text{type } I),\\
\mathfrak{b}_{g,b-1}(v) & = & -\cup u'' & \text{with} & u'' & \subset & u'\cup v' & (\text{type } II),\\
\mathfrak{a}_{g,b}(v) & = & -\cup \overline{u}'' & \text{with} & \overline{u}'' & \subset & \overline{v}'\cup\overline{u}' & (\text{type } II).
\end{array}$$

Let us denote by $M_i = M\cup\partial^0 M\times[0,i]$. We first prolong the vertical maps in the diagram to obtain

$$\begin{tikzcd}
\mathcal{E}^+_{g,b-1}(M(v);\delta(u)) \arrow[r, "\beta_{g,b-1}(M(v);\delta(u))", "-\cup P(\tilde u'')"'] \arrow[d, "\mathfrak{b}_{g,b-1}(v)"', "-\cup u''"] & \mathcal{E}^+_{g,b}(M_1(\overline{v});\overline{\delta}(\overline{u})) \arrow[d, "\mathfrak{a}_{g,b}(v)"', "-\cup\overline{u}''"] \\
\mathcal{E}^+_{g,b}(M;\delta) \arrow[r, "\alpha_{g,b}(M;\delta)", "-\cup P"'] \arrow[d, "\mathfrak{i}_0(\delta)"', "-\cup\delta^0\times[0,2]"] & \mathcal{E}^+_{g+1,b-1}(M_1;\overline{\delta}) \arrow[d, "\mathfrak{i}_1(\overline{\delta})"', "-\cup\overline{\delta}^0\times[1,3]"] \\
\mathcal{E}^+_{g,b}(M_2;\delta+2) \arrow[r, "\alpha_{g,b}(M_2)", "-\cup(P+2)"'] & \mathcal{E}^+_{g+1,b-1}(M_3;\overline{\delta}+2),
\end{tikzcd} \tag{6.2}$$

using the maps $\mathfrak{i}_j(\delta) = -\cup\delta^0\times[j,j+2]$, with $\delta^0\times[j,j+2]\subset\partial^0 M\times[j,j+2]$ and the map $\alpha_{g,b}(M_2) = -\cup(P+2)$, where $P+2\subset\partial^0 M\times[2,3]$ is the cobordism P translated 2 units. The boundary condition $\delta+2$ is $(\delta\setminus\delta^0)\cup(\partial\delta^0\times[0,2])\cup(\delta^0\times\{2\})$. In the diagram we have specified both the name of the map and the cobordism that defines the map (see Figures 4 and 5).

The maps $\mathfrak{i}_0(\delta)$ and $\mathfrak{i}_1(\overline{\delta})$ are homotopy equivalences, and the bottom square commutes up to homotopy. The purpose of enlarging the diagram is to make room to define a map $-\cup Q$ from the upper right corner to the lower left corner.

In (6.2), we have that

$$M_1(\overline{v})\cup\big(\overline{u}\cup\overline{v}\cup\partial^0 M\times[1,2]\big) = M_2,$$

and we want to find a cobordism $Q\subset N := \overline{u}\cup\overline{v}\cup\partial^0 M\times[1,2]$ defining a map

$$-\cup Q\colon \mathcal{E}^+_{g,b}(M_1(\overline{v});\overline{\delta}(\overline{u})) \longrightarrow \mathcal{E}^+_{g,b}(M_2;\delta+2). \tag{6.3}$$

The boundary of N is divided in three parts:

$$\partial N \quad = \quad \partial M_1(\overline{v}) \quad \cup \quad \partial\partial^0 M_1\times[1,2] \quad \cup \quad \partial^0 M\times\{2\}$$

and the boundary condition which Q must satisfy is

$$\xi \quad = \quad \overline{\delta}(\overline{u}) \quad \cup \quad \partial\delta^0\times[1,2] \quad \cup \quad \delta^0\times\{2\}.$$

Lemma 6.1. *If the dimension of M is at least 6, then there is a disjoint union of strips and cylinders Σ', a cobordism $Q\in\mathcal{E}(\Sigma',N;\xi)$, and isotopies*

$$\begin{array}{rclcl}
P(\tilde u)\cup Q & \overset{G}{\simeq} & u''\cup(\delta^0\times[0,2]) & \subset & u\cup v\cup\partial^0 M\times[0,2]\\
Q\cup(P+2) & \overset{H}{\simeq} & \overline{u}''\cup(\overline{\delta}^0\times[1,3]) & \subset & \overline{u}\cup\overline{v}\cup\partial^0 M\times[1,3].
\end{array}$$

relative to the boundaries. If there is a simply connected open subset Y of $\partial^0 M$ with $P\subset Y\times[0,1]$, then the result holds also for manifolds of dimension 5.

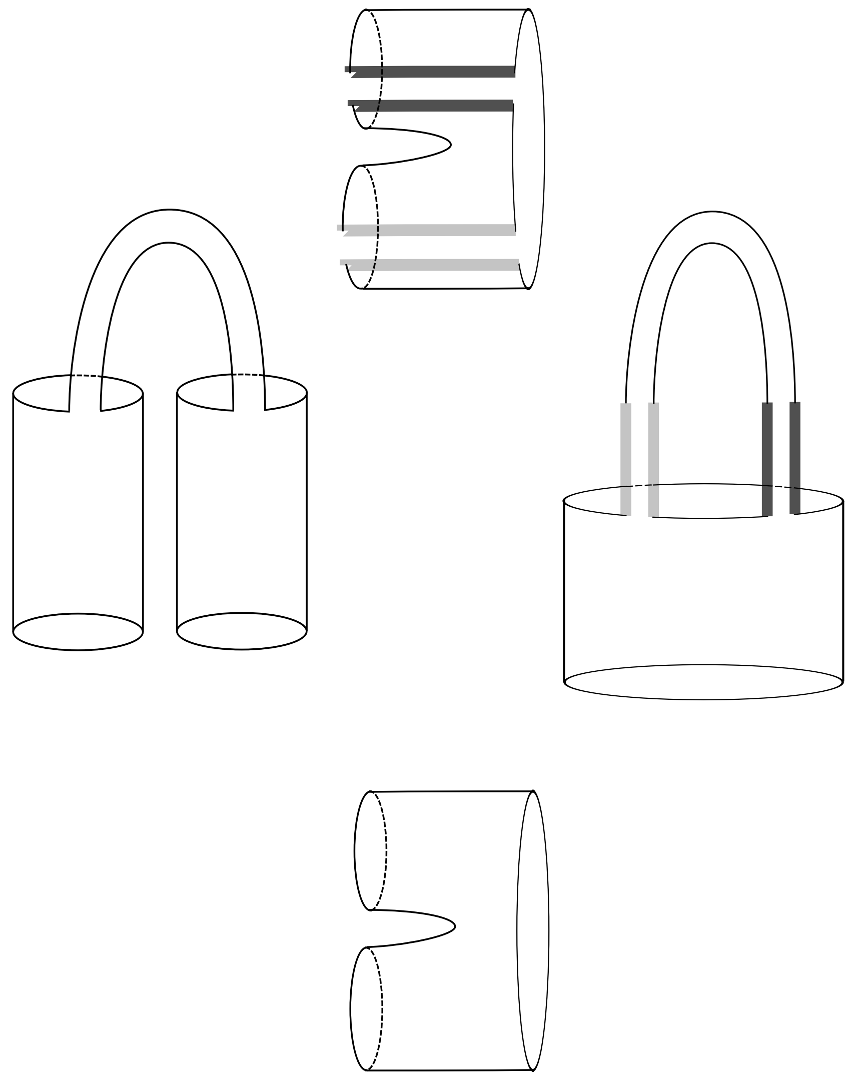


FIGURE 4. The cobordisms in diagram 6.2 when $\delta = \delta^0$. The grey colourings in the boundary show how these pieces of the boundary glue together. The cylinders attached to the other boundary components havenot been drawn.

Proof. Firstly we show that $\mathcal{E}(\Sigma', N; \xi)$ is non-empty. Recall that $\partial\overline{\delta}(\overline{u})^0 = \partial\delta^0(u)^0$ by the assumption made in Section 3.3 (see also Section 2.4). From Construction 5.1, we know that $\overline{\delta}(\overline{u}) \simeq \#_\varphi\delta(u)$. Moreover, $\delta(u) = \#_{u'''}\delta$, and u''' is isotopic in N to η relative to their boundaries, as both are contained in the interior of the thick disc with corners $u \cup v$, hence $\delta(u) = \#_\eta\delta$. In addition, all these isotopies are constant on $\partial\overline{\delta}(\overline{u})^0$. Therefore we conclude that

$$\overline{\delta}(\overline{u}) \simeq \#_\varphi\delta(u) \simeq \#_\varphi\#_\eta\delta \simeq \delta \text{ rel } \partial\overline{\delta}(\overline{u})^0,$$

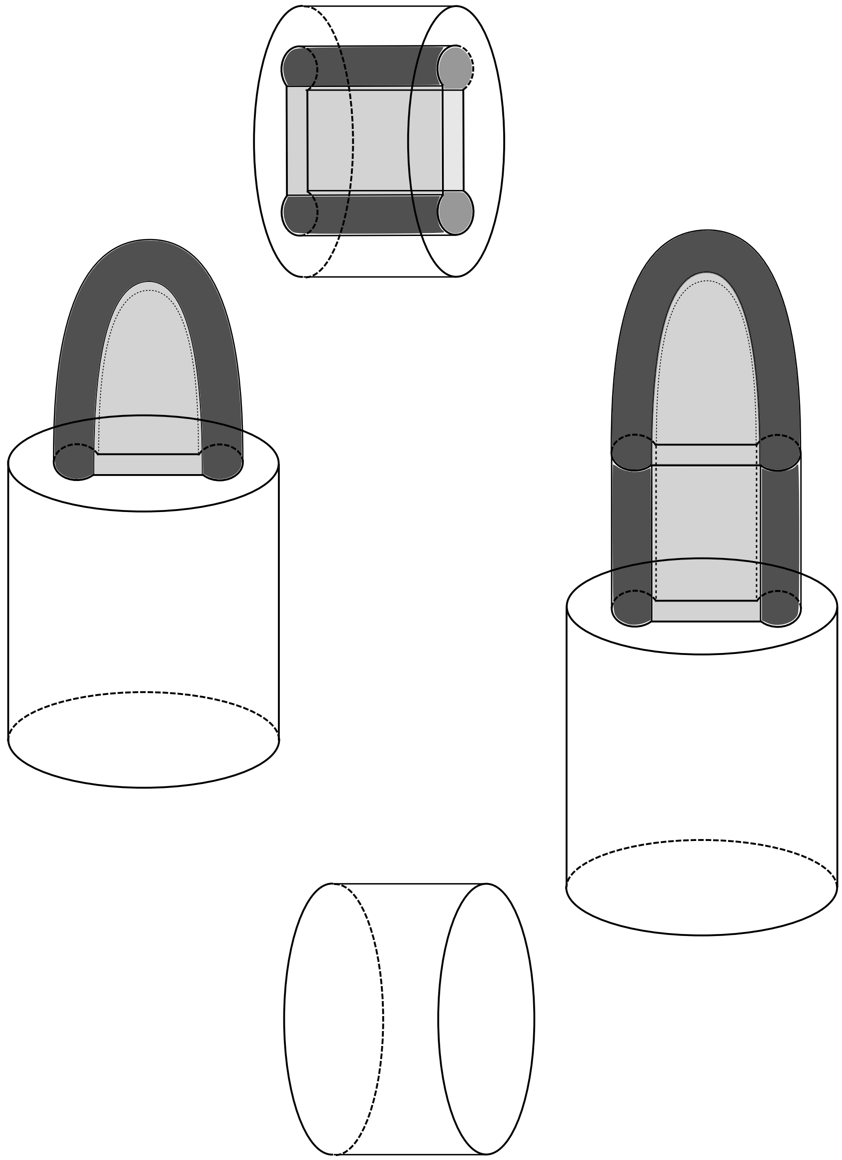


FIGURE 5. The background manifolds in the diagram, in the case when $\partial^0 M$ is a disc. The upper picture should be interpreted as the complement of the coloured figure in the cylinder.

where we have used the natural identification $\partial M(u) \cong \partial M_1(\overline{u})$ and $\partial M \cong \partial M_1$. As the dimension of M is at least 5, this isotopy can be realized as a union of embedded strips $Q \subset N$ whose boundary is ξ and a union of cylinders in the components of δ that are disjoint from u'''. This shows that there exists a $Q \subset N$ giving a map (6.3).

Now, for each triangle of (6.1), we will show how to choose a Q so that the triangle commutes up to homotopy (and these homotopies satisfy certain extra

properties. Note that all the cobordisms in diagram (6.2) contain some part of the product $(\delta^0 \cap (\ell_0 \cup \ell_1)) \times [0,3]$:

$$(\delta^0 \cap (\ell_0 \cup \ell_1)) \times [0,1] \subset P(\tilde{u}), \quad (\delta^0 \cap (\ell_0 \cup \ell_1)) \times [0,2] \subset u'' \cup \delta^0 \times [0,2],$$

$$(\delta^0 \cap (\ell_0 \cup \ell_1)) \times [2,3] \subset P+2, \quad (\delta^0 \cap (\ell_0 \cup \ell_1)) \times [1,3] \subset \overline{u}'' \cup \overline{\delta}^0 \times [1,3].$$

Fix an interval l_0 in $\delta^0(u)\cap\ell_0$ and an interval l_1 in $\delta^0(u)\cap\ell_1$, and let $L_0 = l_0 \times [0,3]$ and $L_1 = l_1 \times [0,3]$, and write L for their union. The extra property we will ensure the homotopies we construct satisfy is that they are constant on L.

The four cobordisms in diagram (6.2) are discs with corners or pairs of pants. Hence the complement of L in each of these cobordisms is a union of discs with corners, that we denote with a superscript L.

Observe first that the inclusion $N \subset u' \cup v' \cup \partial^0 M \times [0,2]$ is an isotopy equivalences, and second that the inclusion $Q \subset P(\tilde{u})^L \cup Q$ is surjective on components (see Figure 7). As each component is a disc (with corners), we deduce that the isotopy type of the discs $P(\tilde{u})^L \cup Q$ is fully determined by the isotopy type of the discs Q. Let us choose a $Q_1 \in \mathcal{E}(\Sigma', N; \xi)$ for which $P(\tilde{u})^L \cup Q_1$ is isotopic to $u'' \cup \delta^0 \times [0,2]$. Similarly, the inclusion $N \subset \overline{u}' \cup \overline{v}' \cup \partial^0 M \times [1,3]$ is an isotopy equivalence and the inclusion $Q \subset Q \cup (P+2)^L$ is surjective on components, hence we may choose a $Q_2 \in \mathcal{E}(\Sigma', N; \xi)$ for which $Q_2 \cup (P+2)^L$ is isotopic to $\overline{u}'' \cup \overline{\delta}^0 \times [1,3]$.

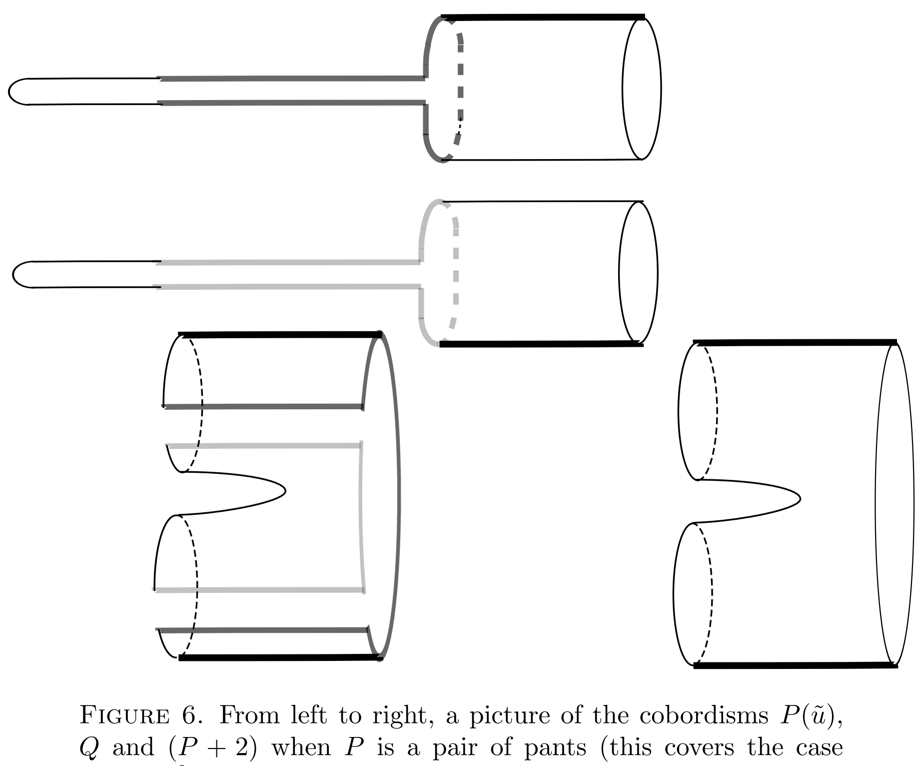


FIGURE 6. From left to right, a picture of the cobordisms $P(\tilde{u})$, Q and $(P+2)$ when P is a pair of pants (this covers the case $\partial M = \partial^0 M$). The thick lines are the strips L, and the grey lines indicate how $P(\tilde{u})$ and Q glue together.

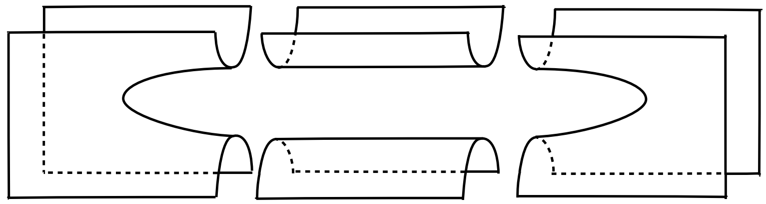


FIGURE 7. An abstract picture (that rounds some corners, for simplicity) of the cobordisms $P(\tilde{u})^L$, Q and $(P+2)^L$. The inclusion $Q \subset P(\tilde{u})^L \cup Q$ is a surjection on π_0, and when restricted to each component of Q, it is an isotopy equivalence onto the component it hits. The same holds for the inclusion $Q \subset Q \cup (P+2)^L$

Finally, we prove that Q_1 is in the same isotopy class as Q_2, hence we may take $Q = Q_1$, finishing the proof of the lemma. For that observe first that the composite cobordisms

$$(u'' \cup \delta^0 \times [0,2])^L \cup (P+2)^L, \qquad P(\tilde{u})^L \cup (\overline{u}'' \cup \overline{\delta}^0 \times [1,3])^L,$$

which are the complements of L of the two ways in diagram 6.2, are isotopy equivalent. Therefore, by the choices made,

$$R_1 := P(\tilde{u})^L \cup Q_1 \cup (P+2)^L, \qquad R_2 := P(\tilde{u})^L \cup Q_2 \cup (P+2)^L$$

are isotopy equivalent and so are the complements $R_1 \setminus \mathring{Q}_1$ and $R_2 \setminus \mathring{Q}_2$. Take now isotopy equivalent parametrizations $f_1, f_2\colon \Sigma_{1,2} \to \partial^0 M \times [0,3]$ of R_1 and R_2 with the same jet d near their boundaries and such that $f_1^{-1}(Q_1) = f_2^{-1}(Q_2) =: S$. In addition fix an isotopy between them. The restriction of these parametrizations to S define a pair of points in $\pi_0(\mathrm{Fib}_{f_{1|\Sigma_{1,2}\setminus S}}(p))$ of the restriction map

$$p\colon \pi_0\mathrm{Emb}(\Sigma_{1,2}, \partial^0 M \times [0,3]; d) \to \pi_0\mathrm{Emb}(\Sigma_{1,2} \setminus S, \partial^0 M \times [0,3]; d_{|\Sigma_{1,2}\setminus S}).$$

These points are mapped to the same point $[f_1] = [f_2]$, and the next Lemma 6.2 implies that they are in the same isotopy class, hence $[f_{1|S}] = [f_{2|S}]$, so $[Q_1] = [Q_2]$, as we wanted.

If there is a simply connected subset $Y \subset \partial^0 M \times [0,1]$ with $P \subset Y$, we may perform the same proof replacing $\partial^0 M$ by Y. $\square$

Lemma 6.2. *Let X be a manifold of dimension at least 6 or a simply connected manifold of dimension 5. Let Σ be an oriented surface, let $S \subset \Sigma$ be a disc and write $\Sigma' := \mathrm{cl}\,(\Sigma \setminus S)$. Let $\overline{f}\colon \Sigma \to X$ be an embedding with jet d. Then, the inclusion of the fiber $\mathrm{Fib}_f(p)$ of the locally trivial fibration $p\colon \mathrm{Emb}(\Sigma, X; d) \to \mathrm{Emb}(\Sigma', X; d_{|\Sigma'})$ over $f_{|\Sigma'}$ is injective on π_0.*

Proof. Let τ be a tubular neighbourhood of $f(\Sigma')$, and let d' be the jet of $f_{|S}\colon S \to X \setminus \tau$. Let $q\colon \mathrm{map}(\Sigma, X; d) \to \mathrm{map}(\Sigma', X; d_{|\Sigma'})$ be the restriction map, which is

also a locally trivial fibration. There is a commutative square

$$\begin{tikzcd}
\pi_0\mathrm{Fib}_f(p) \arrow[rr, "i"] & & \pi_0\mathrm{Emb}(\Sigma, X; d) \arrow[ddd, "k"] \\
\pi_0\mathrm{Emb}(S, X \setminus \tau; d') \arrow[u, "a"] \arrow[d, "b"] & & \\
\pi_0\mathrm{map}(S, X \setminus \tau; d') \arrow[d, "c"] & & \\
\pi_0\mathrm{Fib}_f(q) = \pi_0\mathrm{map}(S, X; d') \arrow[rr, "j"] & & \pi_0\mathrm{map}(\Sigma, X; d).
\end{tikzcd} \tag{6.4}$$

The map a is induced by a homotopy equivalence, and c is a bijection due to the high dimension of X and the fact that $X \setminus \tau$ is homotopy equivalent to the complement in X of a union of submanifolds of dimension 1 (namely, the image of a 1-skeleton of Σ'). In addition, $X \setminus \tau$ remains simply connected if X is simply connected and of dimension 5, hence the map b is injective (and surjective) by Haefliger's theorem [Hae61].

Moreover, the lower map fits in the exact sequence of pointed sets

$$\pi_1\mathrm{map}(\Sigma', X)_f \xrightarrow{y} \pi_0\mathrm{Fib}_f(q) \xrightarrow{j} \pi_0\mathrm{map}(\Sigma, X; d)\ , \tag{6.5}$$

and we claim that the action of $\pi_1\mathrm{map}(\Sigma', X)_f$ on $\pi_0\mathrm{Fib}_f(q)$ is trivial: Given $A \in \pi_0(\mathrm{Fib}_{f_{|'_\Sigma}}(p))$ and $B \in \pi_1(map(\Sigma', X), f|'_\Sigma)$, that is, a map $A : D^2 \to X$ from a disc (possibly with corners) D^2satisfying a boundary condition and a map $B : [0,1] \times \Sigma' \to X$ whose restriction to $\{0,1\} \times \Sigma'$ is $\{0,1\} \times f_{|'_\Sigma}$, the element $y(B, A)$ is obtained by gluing the map $B_{|_{[0,1]\times\partial\Sigma'}}$ to A to get a new map from the disc. As $B_{|_{[0,1]\times\partial\Sigma'}}$ extends to B (by construction), both A and $y(B, A)$ are cobordant, so the two (relative) cycles A and $y(B, A)$ are homologous. But as X is simply connected, the (relative) Hurewicz map is a bijection, so A and $y(B, A)$ are homotopic.

As a consequence, the map j is injective, hence jcb is injective, and so is kia, and since a is a bijection, we deduce that i is injective. □

Composing the map $-\cup Q$ with an inverse of the homotopy equivalence $\mathfrak{i}_0(\delta)$ we obtain a diagonal map for the square (6.1), and the isotopies found in Lemma 6.1 yield the following

Corollary 6.3 (To Lemma 6.1)**.** *In the square* (6.1) *the dashed diagonal map exists and both triangles commute up to homotopy.*

6.2. **Zero in homology.** During this section, if $A \to X$ is a map, we will denote by (X, A) its mapping cone. We will use the letter Σ for "unreduced suspension", and write $CX = [0,1] \times X/\{1\} \times X$.

Lemma 6.4. *If*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "f"] \\
A' \arrow[r, "j"] & X'
\end{tikzcd}$$

is a map of pairs and there is a map $t\colon X \to A'$ making the bottom triangle commute up to a homotopy $H\colon f \simeq jt$, then the induced map between mapping cones $(f,g)\colon (X,A) \to (X',A')$ factors as $(X,A) \xrightarrow{p} CA \cup_i CX \xrightarrow{h} (X',A')$, where p comes from the Puppe sequence. In addition, if there is also a homotopy $G\colon g \simeq ti$, then the composite $CA \cup_i CX \xrightarrow{h} (X',A') \xrightarrow{p'} CA' \cup_j CX'$ is nullhomotopic.

Proof. The map $h\colon CA \cup_i CX \to CA' \cup_j X'$ is given by

$$\begin{array}{rcll} h(a,s) &=& (g(a),s) \in CA' & \text{if} \quad (a,s) \in CA \\ h(b,s) &=& H(b,2s) \in X' & \text{if} \quad (b,s) \in CX \text{ and } 0 \leq s \leq 1/2 \\ h(b,s) &=& (t(b),2s-1) \in CA' & \text{if} \quad (b,s) \in CX \text{ and } 1/2 \leq s \leq 1, \end{array}$$

and it restricts to (f,g) in the mapping cone $CA \cup_i X$, hence $hp = (f,g)$.

For the second part, let $C_{\frac{1}{2}}Y = \{(y,s) \in CY \mid 0 \leq s \leq 1/2\}$, notice that

$$(CA \cup_i CX)/C_{\frac{1}{2}}X \cong \Sigma A \vee \Sigma X,$$

and consider the diagram

$$\begin{tikzcd}
CA \cup CA = \Sigma A \arrow[d, "\mathrm{Id}\cup Ci", "\simeq"'] & & & \\
CA \cup_i CX \arrow[r, "h"] \arrow[d, "\text{collapse } C_{\frac{1}{2}}X"] & CA' \cup_j X' \arrow[r, "p'"] & CA' \cup_j CX' \arrow[r, "\text{collapse } CX'", "\simeq"'] & \Sigma A' \\
(CA \cup_i CX)/C_{\frac{1}{2}}X \cong \Sigma A \vee \Sigma X \arrow[rrr, "\Sigma g \vee \Sigma t"] & & & \Sigma A' \vee \Sigma A' \arrow[u, "\nabla"']
\end{tikzcd}$$

which is easily checked to commute. As ti is homotopic to g, the lower composition is homotopic to $\nabla \circ (\Sigma g \vee \Sigma g) \circ \vee$, i.e. $\Sigma g - \Sigma g$, so is nullhomotopic, as required. □

We now return to (6.1), where we had chosen an arc $u_0 = u \in \mathcal{O}^2_{g,b}(M;\delta)_0$ and a disc $v_0 = v \in \mathcal{D}_{g,b}(M(u_0);\delta(u_0))_0$. Suppose that we are given another arc $u_1 \in \mathcal{O}_{g,b-1}(M(v_0);\delta(u_0))_0$ and another disc $v_1 \in \mathcal{D}_{g-1,b}(M(v_0)(u_1),\delta(u_0))_0$ (We can also consider u_1 to lie in $\mathcal{O}_{g,b}(M;\delta)_0$ and u_0 to lie in $\mathcal{O}_{g,b-1}(M(v_1);\delta(u_1))_0$). The diagram below shows the various maps which can be constructed from these data using Lemma 6.4.

$$\begin{tikzcd}
\mathcal{E}^+_{g,b-1}(M(v_0)) \arrow[r] \arrow[d, "\mathfrak{b}_{g,b-1}(v_0)"'] \arrow[dr, phantom, "(1)"] & \mathcal{E}^+_{g,b}(M(v_0)) \arrow[r] \arrow[d, "\mathfrak{a}_{g,b}(v_0)"] & \beta_{g,b-1}(M(v_0)) \arrow[r, "p"] \arrow[d] & \Sigma\mathcal{E}^+_{g,b-1}(M(v_0)) \arrow[dl, "h"] \\
\mathcal{E}^+_{g,b}(M) \arrow[r] & \mathcal{E}^+_{g+1,b-1}(M) \arrow[r] & \alpha_{g,b}(M) & \\
\mathcal{E}^+_{g,b-1}(M(v_1)) \arrow[r] \arrow[u, "\mathfrak{b}_{g,b-1}(v_1)"] \arrow[ur, phantom, "(2)"] & \mathcal{E}^+_{g,b}(M(v_1)) \arrow[r] \arrow[u, "\mathfrak{a}_{g,b}(v_1)"'] & \beta_{g,b-1}(M(v_1)) \arrow[r, "p'"] \arrow[u] & \Sigma\mathcal{E}^+_{g,b-1}(M(v_1)) \arrow[ul, "h'"'] \\
\mathcal{E}^+_{g-1,b}(M(v_0,v_1)) \arrow[r] \arrow[u, "\mathfrak{a}_{g-1,b}(v_0)"] \arrow[ur, phantom, "(3)"] & \mathcal{E}^+_{g,b-1}(M(v_0,v_1)) \arrow[r] \arrow[u, "\mathfrak{b}_{g,b-1}(v_0)"'] & \alpha_{g-1,b}(M(v_0,v_1)) \arrow[r] \arrow[u] & \Sigma\mathcal{E}^+_{g-1,b}(M(v_0,v_1)) \arrow[ul, "h''"'] \arrow[uuu, bend right, "\Sigma\mathfrak{a}_{g-1,b}(v_0,v_1)"']
\end{tikzcd}$$

In this diagram, the second line is the Puppe sequence for the map $\alpha_{g,b}(M;\delta,\overline{\delta})$, and the first and third lines are the Puppe sequences for the approximate augmentations corresponding to the data (u_0,v_0) and (u_1,v_1) respectively. The fourth line is the Puppe sequence for the approximate augmentation obtained by using the data

(u_0, v_0) on the map $\beta_{g,b-1}(M(v_1))$. Importantly, it may also be considered to be the Puppe sequence for the approximate augmentation obtained by using the data (u_1, v_1) on the map $\beta_{g,b-1}(M(v_0))$.

We use Corollary 6.3 in order to provide a diagonal map and isotopies in the square (3), which gives the map h'' such that $p' \circ h''$ is nullhomotopic. We use the same diagonal map and isotopies in the square (1) to obtain the map h, because in square (1) the problem of finding such data is the same as in square (3), but without the requirement that the isotopies have to fix the cylinders $\tilde{u}'_1$. We choose any diagonal map and isotopies for the square (2), to obtain the map h'. After doing that, it follows from the definition of the map h that

Lemma 6.5. *The maps*

$$(\mathfrak{a}_{g,b}(v_1), \mathfrak{b}_{g,b-1}(v_1)) \circ h'' \quad \textit{and} \quad h \circ \Sigma\mathfrak{a}_{g-1,b}(v_0, v_1)$$

from $\Sigma\mathcal{E}^+_{g-1,b}(M(v_0, v_1))$ *to* $\alpha_{g,b}(M)$ *are homotopic.*

Proof. The map $\Sigma\mathfrak{a}_{g-1,b}(v_0, v_1)$ is the suspension of the map that glues back the strip u''_1 corresponding to v_1. The map $(\mathfrak{a}_{g,b}(v_1), \mathfrak{b}_{g,b-1}(v_1))$ glues back the strips u''_1 and $\overline{u}''_1$. The homotopies in square (3) fix these strips by very definition, and we have chosen the homotopies in (1) to be the ones in square (3), so they fix the strips too. Therefore the manifolds in which the cobordisms that define h and the cobordisms that define $\Sigma\mathfrak{a}_{g-1,b}(v_0, v_1)$ live are disjoint. The maps h and h'' are obtained by gluing to the spaces of manifolds the same cobordisms and performing the same isotopies on them. □

Proposition 6.6. *Let M be simply connected and of dimension at least* 5. *If the dimension is* 5, *then we assume that all the stabilisation maps in what follows are induced by pairs of pants that are contractible in $\partial^0 M \times I$. If X_{g-1} holds, then the map $(\beta_{g,b-1}(M(v_0))) \to (\alpha_{g,b}(M))$ induces the zero map in homology degrees $* \leq \frac{2g+2}{3}$. If Y_{g-1} holds, then the map $(\alpha_{g,b-1}(M(u,v))) \to (\beta_{g,b}(M))$ induces the zero map in homology degrees $* \leq \frac{2g+1}{3}$.*

Proof. There are homotopies

$$h \circ \Sigma\mathfrak{a}_{g-1,b}(v_0, v_1) \simeq (\mathfrak{a}_{g,b}(v_1), \mathfrak{b}_{g,b-1}(v_1)) \circ h'' \simeq h' \circ p' \circ h'' \simeq *$$

by applying Lemmas 6.5 and 6.4. Since X_{g-1} holds, the map $\Sigma\mathfrak{a}_{g-1,b}(v_0, v_1)$ induces an epimorphism in homology degrees $* \leq \frac{2(g-1)+1}{3} + 1$ (although the map $\mathfrak{a}_{g-1,b}(v_0, v_1)$ is not a map of type $\alpha_{g,b}(M(v_0, v_1)$, it is isotopic to such a map after rounding the corners of M), hence h must induce the zero homomorphism in those degrees, and so must hp. The second part is proven similarly, by rewriting all of this section in the analogous way. □

The following finishes the proof of parts (v) and (vi) of Proposition 4.3.

Corollary 6.7. *Let M be a simply connected manifold of dimension at least* 5. *If the dimension of M is* 5 *we assume in addition that the pairs of pants defining the stabilisation maps are contractible in $\partial^0 M \times [0,1]$. If X_{g-1} and G_g hold, then the map*

$$(\beta_{g,b-1}(M(u_0))) \longrightarrow (\alpha_{g,b}(M))$$

induces the zero homomorphism in homology degrees $k \leq \frac{2g+2}{3}$. If Y_{g-1} and F_{g-1} hold, then the map

$$(\alpha_{g-1,b+1}(M(u_0))) \longrightarrow (\beta_{g,b}(M))$$

induces the zero homomorphism in homology degrees $k \leq \frac{2g+1}{3}$.

Proof. In the first case, by the previous proposition we have seen that if X_{g-1} holds, then the composition

$$(\beta_{g,b-1}(M(v))) \longrightarrow (\beta_{g,b-1}(M(u_0))) \longrightarrow (\alpha_{g,b}(M))$$

induces the zero homomorphism in degrees $* \leq \frac{2g+2}{3}$, while in Proposition 5.7 we have proven that if G_g holds, then the left arrow is an epimorphism in degrees $* \leq \frac{2g+3}{3}$. Thus the composition is zero in the range of degrees claimed.

The second case is completely analogous. □

7. Homological stability for closed surfaces

7.1. **Resolutions.** Consider the space $\mathcal{E}^+_{g,b}(M;\delta)$, and let $\ell \subset \partial^0 M$ be a subset diffeomorphic to a ball disjoint from δ. There is a semi-simplicial space $\mathcal{P}_{g,b}(M;\delta,\ell)_\bullet$ whose i-simplices are tuples $(W, p_0, \dots, p_i)$, where $p_j = (p'_j, p''_j, p'''_j)$ and

(i) $W \in \mathcal{E}^+_{g,b}(M;\delta)$,
(ii) $p'''_j \colon ([0,1], \{1/2\}) \to (M, W)$ is an embedding with $p'''_j(0) \in \ell$ and $p'''_j(1) \in \mathring{M}$,
(iii) (p'_j, p''_j) is a closed tubular neighbourhood of $p'''_j([0,1])$ in the pair (M, W).
(iv) the neighbourhoods $p'_0, \dots, p'_i$ are disjoint.

The jth face map forgets p_j and there is an augmentation map to $\mathcal{E}^+_{g,b}(M;\delta)$ that forgets all the p_j. We topologise the space of i-simplices as a subset of $\mathcal{E}^+_{g,b}(M;\delta) \times \overline{\mathrm{T}}\mathrm{Emb}(I \times [i], M; q, q_N)$, where

$$q(x) = q_N(x) = \ell \text{ if } x = 0, \qquad q(x) = q_N(x) = M \text{ if } x \neq 0.$$

Proposition 7.1. *If M is connected and of dimension at least 3, the semi-simplicial space $\mathcal{P}_{g,b}(M;\delta)_\bullet$ is a resolution of $\mathcal{E}^+_{g,b}(M;\delta)$.*

Proof. The space $\mathcal{E}^+_{g,b}(M;\delta)$ is $\mathrm{Diff}_\partial(M)$-locally retractile by Lemma 2.13, hence is also $\mathrm{Diff}(M;\delta,\ell)$-locally retractile, and the augmentation map ϵ_i is $\mathrm{Diff}(M;\delta,\ell)$-equivariant for all i, therefore it is also a locally trivial fibration by Lemma 2.8. As a consequence, the semi-simplicial fiber $\mathrm{Fib}_W(\epsilon_\bullet)$ is homotopy equivalent to the homotopy fiber of $|\epsilon_\bullet|$ by Criterion 2.21. The space of i-simplices of the semi-simplicial fiber is $\overline{\mathrm{T}}\mathrm{Emb}((I \times [i], \{1/2\} \times [i]), (M, W); q, q_N)$. Let us define the semi-simplicial space $P(W, M)_\bullet$ whose space of i-simplices is

$$\mathrm{Emb}((I \times [i], \{1/2\} \times [i]), (M, W); q),$$

and the face maps are given by forgetting embeddings. Forgetting the tubular neighbourhoods gives a map

$$r_\bullet \colon \mathrm{Fib}_W(\epsilon_\bullet) \longrightarrow P(W, M)_\bullet$$

that is levelwise $\mathrm{Diff}(M; W, \ell)$-equivariant onto the space $P(W, M)_\bullet$, which is levelwise $\mathrm{Diff}(M; W, \ell)$-locally retractile by Corollary 2.18, hence this map is a levelwise fibration by lemma 2.8. The fiber over an i-simplex $\mathbf{p} = (p'''_0, \dots, p'''_i)$ is the space $\overline{\mathrm{Tub}}(\mathbf{p}'''(I), (M, W); q_N)$, which is contractible by Lemma 2.4.

The semi-simplicial space $P(W, M)_\bullet$ is a topological flag complex, and we will apply Criterion 2.22 to prove that it is contractible. As M is connected, for each

tuple $(W, p_0'''), \dots, (W, p_{i-1}''')$ of 0-simplices over a surface W, there is another 0-simplex (W, p_i''') over W orthogonal to them all, by general position. Hence it is contractible by Criterion 2.22. □

Let $B_i(M;\ell)$ be the set of tuples $(p_0, \dots, p_i)$ with $p_j = (p_j', p_j'', p_j''')$ where p_j''' is an embedding of an interval in M, p_j' is a tubular neighbourhood of p_j'' in $M$and p_j'' is the restriction of p_j' to some vector subspace $L_j \subset \overline{\mathrm{N}}_M p_j'''(1/2)$ of dimension 2. Moreover, we require that p_j' is disjoint from p_k'. This space is in canonical bijection with $\overline{\mathrm{T}}\mathrm{Emb}_{2,\{1/2\}\times[i]}(I \times [i], M; q, q_N)$, and we use this bijection to topologize it.

There is a map

$$\mathcal{P}_{g,b}(M;\delta,\ell)_i \longrightarrow B_i(M;\ell)$$

that sends a tuple $(W, p_0, \dots, p_i)$ to the tuple $(p_0, \dots, p_i)$.

Proposition 7.2. *For a point $\mathbf{p} \in B_i(M;\ell)$, with the notation of Section 2.4, there is a homotopy fibre sequence*

$$\mathcal{E}^+_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p})) \longrightarrow \mathcal{P}_{g,b}(M;\delta)_i \longrightarrow B_i(M;\ell)$$

Proof. The map is $\mathrm{Diff}(M;\delta,\ell)$-equivariant and the space $B_i(M;\ell)$ is $\mathrm{Diff}(M;\delta,\ell)$-locally retractile by Lemma 2.19, hence this map is a locally trivial fibration by Lemma 2.8. The fiber over a point $\mathbf{p}$ is the space of surfaces W in M that meet the tubular neighbourhoods p_j' in the image of p_j''. This space is canonically homeomorphic to the space $\mathcal{E}^+_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p}))$. □

7.2. **Stabilisation maps between resolutions.** In this section we will show how to extend the stabilisation map $\gamma_{g,b}(M;\delta)$ to a map between resolutions.

$$\begin{tikzcd}
\mathcal{P}_{g,b}(M;\delta,\ell)_i \arrow[r, dashed] \arrow[d] & \mathcal{P}_{g,b-1}(M_1;\overline{\delta},\overline{\ell})_i \arrow[d] \\
\mathcal{E}^+_{g,b}(M,\delta) \arrow[r, "\gamma_{g,b}(M_1;\delta,\overline{\delta})"] & \mathcal{E}^+_{g,b-1}(M_1,\overline{\delta})
\end{tikzcd}$$

To define the maps $\gamma_{g,b}(M;\delta,\overline{\delta})\colon \mathcal{E}^+_{g,b}(M;\delta) \to \mathcal{E}^+_{g,b-1}(M_1,\overline{\delta})$ we joined each surface with a cobordism P in $\partial^0 M \times I$. We will assume, without loss of generality, that

(i) $\overline{\ell} = \ell \times \{1\}$,
(ii) $(\ell \times I) \cap P = \emptyset$.

As in previous constructions, we define $\tilde{p}_j' = \partial^0 p_j' \times I$ and $\overline{p}_j' = p_j' \cup \tilde{p}_j'$ and similarly $\tilde{p}_j'$ and $\overline{p}_j'$. There is a map $\gamma_{g,b}(M;\delta,\overline{\delta})_i$ making the diagram commute, that sends a tuple $(W, \mathbf{p})$ to the tuple $(W \cup P, \overline{\mathbf{p}})$. These maps commute with the face maps and with the augmentation maps, so they define a map of semi-simplicial spaces (the *resolution of* $\gamma_{g,b}(M;\delta,\overline{\delta})$)

$$\gamma_{g,b}(M;\delta,\overline{\delta})_\bullet\colon \mathcal{P}_{g,b}(M;\delta)_\bullet \longrightarrow \mathcal{P}_{g,b-1}(M_1;\overline{\delta})_\bullet$$

extending $\gamma_{g,b}(M;\delta,\overline{\delta})$.

Corollary 7.3 (To Proposition 7.1). *The pair $(\gamma_{g,b}(M)_\bullet, \delta, \overline{\delta})$ together with the natural augmentation to the pair $\gamma_{g,b}(M;\delta,\overline{\delta})$ is a resolution.*

$$\begin{tikzcd}
\mathcal{P}_{g,b}(M;\delta)_i \arrow[r, "\gamma_{g,b}(M;\delta,\overline{\delta})_i"] \arrow[d] & \mathcal{P}_{g,b-1}(M;\overline{\delta})_i \arrow[d] \\
B_i(M;\ell) \arrow[r, "\mathbf{p}\mapsto\overline{\mathbf{p}}"] & B_i(M_1,\ell)
\end{tikzcd}$$

is an extension of the homotopy equivalence $B_i(M;\ell) \to B_i(M_1;\overline{\ell})$, hence we obtain a well-defined map on the homotopy fibres over the points $\mathbf{p}$ and $\overline{\mathbf{p}}$ of the fibrations of Proposition 7.2,

$$\mathcal{E}^+_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p})) \longrightarrow \mathcal{E}^+_{g,b+i}(M(\overline{\mathbf{p}});\overline{\delta}(\overline{\mathbf{p}}))$$

obtained by gluing the cobordism P to each surface. This is a map of type $\gamma_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p}),\overline{\delta}(\overline{\mathbf{p}}))$.

Corollary 7.4 (To Proposition 7.2)**.** *There is a relative homotopy fibre sequence*

$$(\gamma_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p}),\overline{\delta}(\overline{\mathbf{p}}))) \longrightarrow (\gamma_{g,b}(M;\delta,\overline{\delta})_i) \longrightarrow B_i(M;\ell).$$

7.3. **Homological stability.**

Remark 7.5. Consider a stabilisation map $\gamma_{g,b}(M;\delta,\overline{\delta})$, which is given by closing off one of the boundaries, b, of δ (which must necessarily be nullhomotopic in $\partial^0 M$). If δ has another boundary component, b', in the same component of $\partial^0 M$ as b, then there exists a stabilisation map $\beta_{g,b-1}(M;\delta_0,\delta)$ creating the boundaries b and b'. In this case we may enlarge collars, and we have the composition

$$\mathcal{E}^+_{g,b-1}(M;\delta_0) \xrightarrow{\beta_{g,b-1}(M;\delta_0,\delta)} \mathcal{E}^+_{g,b}(M_1;\delta) \xrightarrow{\gamma_{g,b}(M_1;\delta,\overline{\delta})} \mathcal{E}^+_{g,b-1}(M_2;\overline{\delta})$$

which is homotopic to a stabilisation map which takes the union with a cylinder inside $\partial^0 M \times [0,2]$. This map may not be homotopic to the identity—the cylinder may be embedded in a non-trivial way—but it is a homotopy equivalence (as we may find a inverse cylinder), so the map $\gamma_{g,b}(M_1;\delta,\overline{\delta})$ is split surjective on homology. By the same argument, any map $\beta_{g,b}(M_1;\delta,\overline{\delta})$ which creates a boundary which is nullhomotopic in $\partial^0 M$ is split injective on homology.

The following proposition finishes the proof of Theorem 1.3.

Proposition 7.6. *Let M be a simply connected manifold of dimension at least 5 with non-empty boundary, and δ be a boundary condition. Then for any boundary condition $\overline{\delta}$,*

(i) *for any map $\gamma_{g,b}(M;\delta,\overline{\delta})$ we have $H_*(\gamma_{g,b}(M;\delta,\overline{\delta})) = 0$ for all $* \leq \frac{2g+3}{3}$,*
(ii) *any map $\beta_{g,b}(M;\delta,\overline{\delta})$ with one of the newly created components of $\overline{\delta}$ is contractible in $\partial^0 M$ induces a monomorphism in all homology degrees,*
(iii) *any map $\gamma_{g,b}(M;\delta,\overline{\delta})$ for which there is another component of δ in the same component of $\partial^0 M$ as the one one which is closed induces an epimorphism in all homology degrees.*

Proof. We have already shown the last two statements above. Regarding the first statement, first suppose that there is another component of δ in the same component of $\partial^0 M$ as the one one which is closed by $\gamma_{g,b}(M;\delta,\overline{\delta})$, and choose a $\beta_{g,b-1}(M;\delta_0,\delta)$ as in Remark 7.5. By Proposition 4.1, we know that $\beta_{g,b-1}(M;\delta_0,\delta)$ induces an epimorphism in homology degrees $* \leq \frac{2g}{3}$, and it also induces a monomorphism

in all degrees: thus it induces an isomorphism in degrees $* \leq \frac{2g}{3}$. As $\gamma_{g,b}(M;\delta,\overline{\delta})$ is a left inverse to it, this also induces an isomorphism in these degrees. Hence $H_*(\gamma_{g,b}(M;\delta,\overline{\delta})) = 0$ for $* \leq \frac{2g+3}{3}$, as $\gamma_{g,b}(M;\delta,\overline{\delta})$ induces an epimorphism in all degrees.

Now suppose that there is not an additional component of δ in the same component of $\partial^0 M$ as the one which is closed by $\gamma_{g,b}(M;\delta,\overline{\delta})$. We choose a ball $\ell \subset \partial^0 M$ and form the resolution of $\gamma_{g,b}(M;\delta,\overline{\delta})$ given by Corollary 7.3. Using Corollary 7.4 to identify the space of i-simplices in this resolution, the pair $(\gamma_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p}))$ is a map of type γ for surfaces with (after rounding the corners of M) at least $(i+1)$ extra boundary components of $\delta(\mathbf{p})$ in the component of ∂M containing the boundary which is closed off, so the discussion above applies and shows that $H_*(\gamma_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p})) = 0$ for $* \leq \frac{2g+3}{3}$. Applying the second result of Criterion 2.24 to this resolution gives the result. □

8. Stable homology of the space of surfaces in a manifold with boundary

In this section we prove Theorem 1.5 in the case where M has non-empty boundary, apart from the proof of a proposition (Proposition 8.9) which we defer to the next section. We will show how to deduce Theorem 1.2 for manifolds without boundary in Section 10. During the rest of the paper we will only work with manifolds and manifolds with boundary (but not general manifolds with corners).

8.1. **Spaces of manifolds and scanning maps.** Let M be a smooth manifold of dimension d, possibly with boundary. Recall from [GRW10, Definition 2.1] and [RW11, Section 3] that the set $\Psi(M)$ of all smooth oriented 2-dimensional submanifolds of $\mathring{M}$ which are closed as subsets of M can be endowed with a topology.

More generally, for any real vector space V we can define the space $\Psi(V)$ of smooth oriented 2-dimensional manifolds in V. If $\langle -,- \rangle$ is an inner product on V, there is a space $\mathcal{S}(V)$ as in Definition 1.1, and an inclusion

$$i : \mathcal{S}(V) \longrightarrow \Psi(V) \tag{8.1}$$

given by sending a pair $(L \in \mathrm{Gr}_2^+(V), v \in L^\perp)$ to the oriented 2-manifold $v+L \subset V$, and sending the point at infinity to the empty manifold.

Proposition 8.1 ([GRW10])**.** *The inclusion i is a weak homotopy equivalence.*

Let $\mathfrak{g}$ be a Riemannian metric on M, not necessarily complete. There is an associated partially defined exponential map $\exp\colon TM \dashrightarrow M$. The *injectivity radius* of $\mathfrak{g}$ at $p \in M$ is the supremum of the real numbers $r \in (0,\infty)$ such that exp is defined on T_pM on vectors of length $< r$, and exp is injective when restricted to the open ball of radius r in T_pM.

Let $a\colon M \to (0,\infty)$ be a smooth map which at each point is strictly less than the injectivity radius of the metric $\mathfrak{g}$ at that point: such functions exist by a partition of unity argument. If V is an inner product space, define an endomorphism h of V by $v \mapsto \left(\frac{1}{\pi}\arctan||v||\right) v$. Let $\exp_a\colon TM \to M$ be the composition of the endomorphism of TM given by $v \mapsto a(p)h(v)$ if $v \in T_pM$, and the exponential map.

Let $\Psi(TM)$ denote the space of pairs (p,W) with $p \in M$ and $W \in \Psi(T_pM)$, i.e. the space obtained by performing the construction $\Psi(-)$ fibrewise to TM. There

is a map

$$\Psi(M) \times M \longrightarrow \Psi(TM),$$

given by $(W, p) \mapsto ((\exp_a|_{T_pM})^{-1}(W) \subset T_pM)$, whose adjoint

$$s_a \colon \Psi(M) \longrightarrow \Gamma(\Psi(TM) \to M),$$

a map to the space of sections of the bundle $\Psi(TM) \to M$, is called *non-affine scanning map*.

Proposition 8.2 ([RW11]). *If M has no compact components, the non-affine scanning map s_a is a weak homotopy equivalence.*

On the other hand, let $\mathcal{S}(TM)$ denote the result of performing the construction $\mathcal{S}(-)$ fiberwise to the tangent bundle of M. Let $\Psi^\nu(M)$ denote the space of pairs (W, t) of a submanifold $W \in \Psi(M)$ and a map $t \colon W \to (0, \infty)$, such that the exponential map restricted to the subspace

$$\nu_t(W) := \{(w, v) \in NW \mid ||v|| < t(w)\} \subset NW$$

is an embedding. There is a map

$$\Psi(M)^\nu \times M \longrightarrow \mathcal{S}(TM)$$

given by

$$(W, t, p) \longmapsto \begin{cases} \infty & \text{if } p \notin \exp(\nu_t(W)) \\ (D(\exp|_{T_wM}))(T_wW \perp v) \subset T_pM & \text{if } p = \exp(w, v) \text{ for } (w, v) \in \nu_t(W), \end{cases}$$

where we consider the oriented 2-plane T_wW and vector v as lying inside $T_v(T_wM)$ using the canonical isomorphism $T_v(T_wM) \cong T_wM$, and then apply the linear isomorphism $D(\exp|_{T_wM}) : T_v(T_wM) \longrightarrow T_pM$. The adjoint to this map

$$s \colon \Psi^\nu(M) \longrightarrow \Gamma(\mathcal{S}(TM) \to M)$$

is called the *scanning map*.

Because the inclusion (8.1) is $O(n)$ equivariant, it follows that the inclusion

$$i \colon \Gamma(\mathcal{S}(TM) \to M) \longrightarrow \Gamma(\Psi(TM) \to M)$$

is a weak homotopy equivalence. The projection $\pi \colon \Psi^\nu(M) \to \Psi(M)$ is also a weak homotopy equivalence, by Lemma 2.4. The following proposition shows that the scanning map is also a weak homotopy equivalence if M has no compact components.

Proposition 8.3. *The square*

$$\begin{tikzcd} \Gamma(\mathcal{S}(TM) \to M) \arrow[r] & \Gamma(\Psi(TM) \to M) \\ \Psi^\nu(M) \arrow[u, "s"] \arrow[r, "\pi"] & \Psi(M) \arrow[u, "s_a"'] \end{tikzcd}$$

commutes up to homotopy, so if M has no compact components then

$$s \colon \Psi^\nu(M) \longrightarrow \Gamma(\mathcal{S}(TM) \to M)$$

is a weak homotopy equivalence.

Proof. For $(V, \langle -,- \rangle)$ an inner product space, let us define an auxiliary subspace

$$\widetilde{\Psi}(V) = \{W \in \Psi(V) \mid W \text{ is empty or has a unique closest point to the origin}\}$$

of $\Psi(V)$, which is again natural in $(V, \langle -,- \rangle)$. This subspace contains $\mathcal{S}(V)$, and we wish to show that the inclusion $\mathcal{S}(V) \hookrightarrow \widetilde{\Psi}(V)$ is a homotopy equivalence. Let $\widetilde{\Psi}(V)_0$ be the subspace consisting of the non-empty manifolds: there is a continuous function

$$c\colon \widetilde{\Psi}(V)_0 \longrightarrow V$$

which picks out the unique closest point to the origin (this extends to a continuous function $c^+ \colon \widetilde{\Psi}(V) \to V^+$ by sending the empty manifold to the point at infinity).

We define a homotopy $H_V \colon [1,\infty] \times \widetilde{\Psi}(V) \to \widetilde{\Psi}(V)$ by the rule

$$(t, W) \longmapsto \begin{cases} c(W) + t \cdot (W - c(W)) & \text{if } t < \infty \text{ and } W \text{ is non-empty} \\ c(W) + T_{c(W)}W & \text{if } t = \infty \text{ and } W \text{ is non-empty}, \\ \emptyset & \text{if } W \text{ is empty.} \end{cases}$$

which may be checked to be continuous in the topology of [GRW10, Definition 2.1].

The above discussion is completely natural in the inner product space $(V, \langle -,- \rangle)$, so we may apply it fibrewise to any vector bundle with metric. In particular, the homotopies H_{T_pM} fit together to give a deformation retraction H of $\widetilde{\Psi}(TM)$ onto $\mathcal{S}(M)$. The map $s_a \circ \pi$ lands in $\Gamma(\widetilde{\Psi}(TM) \to M)$, and applying the deformation retraction H gives a homotopic map $H(\infty, -) \circ s_a \circ \pi$, which by inspection is equal to the map s. $\square$

8.2. **Scanning maps with boundary conditions.** We will also often need scanning maps when M has boundary, and surfaces are required to satisfy a boundary condition, as in Section 2.2. We formalise this as follows.

Let M be a manifold with boundary, $c\colon (-1, 0] \times \partial M \to M$ be a collar, and $\xi \subset \partial M$ be a compact oriented 1-manifold. Write $M(\infty) = M \cup_{\partial M} ([0,\infty) \times \partial M)$ for the manifold obtained by attaching an infinite collar to M. We let

$$\Psi(M;\xi) := \{W \in \Psi(M(\infty)) \mid W \cap ((-1,\infty) \times \partial M) = (-1,\infty) \times \xi\}.$$

Choosing a Riemannian metric $\mathfrak{g}$ on $M(\infty)$ (which is a product on $(-1,\infty) \times \partial M$) and a function a as above, we obtain a scanning map s_a for $M(\infty)$. If the function a is chosen so that

$$\exp_a(TM(\infty)|_{[0,\infty)\times\partial M}) \subset (-1,\infty) \times \partial M,$$

then for $W \in \Psi(M;\xi)$ the section $s_a(W)$ is a product when restricted to $[0,\infty) \times \partial M$, and is independent of W. By slight abuse of notation we call this product section $s_a([0,\infty) \times \xi)$, and write $\Gamma(\Psi(TM) \to M; s_a(\xi))$ for the space of sections of $\Psi(TM) \to M$ which agree with $s_a([0,\infty) \times \xi)|_{\partial M}$ over ∂M. In this case there is a scanning map

$$s_a \colon \Psi(M;\xi) \longrightarrow \Gamma(\Psi(TM) \to M; s_a(\xi)),$$

by construction. As in Proposition 8.2, if M has no compact components then this scanning map is a weak homotopy equivalence.

8.3. **Adding tails to M.** Suppose that M is a *compact* manifold with collared boundary, and let $N, L \subset \partial M$ be open codimension 0 submanifolds, with L diffeomorphic to a ball.

Definition 8.4. We define the following subspaces of $\partial M \times [0, \infty)$:

$$N_{[a,b]} := N \times [a, b], \qquad L_{[a,b]} := L \times [a, b],$$

and we also write $N_{[0,\infty)} = N \times [0, \infty)$ and $N_a = N_{[a,a]}$, and similarly for L.

We then write

$$M_{a,b} := M \cup N_{[0,a]} \cup L_{[0,b]},$$

and let $M_{a,\infty}$, $M_{\infty,b}$ or $M_{\infty,\infty}$ have their obvious meaning.

Note that the boundary component $N_a \subset M_{a,b}$ has a canonical collar, inside $((-1, 0] \times \partial M) \cup_{\partial M} (N \times [0, \infty))$; similarly for the boundary component $L_b \subset M_{a,b}$.

For $\delta \subset N$ and $\xi \subset L$ compact oriented 1-manifolds, we define

$$\Psi(M_{a,b}; \delta, \xi) := \Psi(M_{a,b}; \delta \cup \xi) \subset \Psi(M_{\infty,\infty}),$$

as in Section 8.2. A careful examination of the topology of $\Psi(M_{\infty,\infty})$ shows that $\Psi(M_{a,b}; \delta, \xi)$ is homeomorphic to the disjoint union $\coprod_{[\Sigma]} \mathcal{E}(\Sigma, M_{a,b}; \delta \cup \xi)$ where $[\Sigma]$ runs along a set of compact oriented surfaces with boundary diffeomorphic to $\delta \cup \xi$, one in each diffeomorphism class.

Restricting the scanning map

$$s_a\colon \Psi(M_{a,b}; \delta, \xi) \to \Gamma_c(\mathcal{S}(TM_{a,b}) \to M_{a,b}; s_a(\delta), s_a(\xi))$$

to the subspace of connected genus g surfaces gives a map

$$\mathscr{S}_{g,c}(\delta, \xi)\colon \mathcal{E}_{g,c}(M_{a,b}; \delta, \xi) \longrightarrow \Gamma_c(\mathcal{S}(TM_{a,b}) \to M_{a,b}; s_a(\delta), s_a(\xi)). \tag{8.2}$$

8.4. **Semi-simplicial models.** In order to show that the map (8.2) induces an isomorphism on homology in a range of degrees, we will pass through certain auxiliary semi-simplicial spaces.

Definition 8.5. For $\xi \subset L_b$ a boundary condition, let $D(M_{\infty,b}; \xi)_p$ be the set of tuples $(a_0, a_1, \ldots, a_p, W)$ where

(i) $0 < a_0 < a_1 < \cdots < a_p$ are real numbers,
(ii) $W \in \Psi(M_{\infty,b}; \xi)$ is a surface satisfying the boundary condition ξ, and the a_i are regular values for the projection $p_W\colon W \cap N_{[0,\infty)} \to [0, \infty)$.

We give it the subspace topology from $(\mathbb{R}^\delta)^{p+1} \times \Psi(M_{\infty,b}; \xi)$. The collection of all the spaces $D(M_{\infty,b}; \xi)_p$ for $p \geq 0$ forms a semi-simplicial space, where the jth face map is given by forgetting a_j, and it is augmented over $\Psi(M_{\infty,b}; \xi)$.

Definition 8.6. Let $D(N_{(0,\infty)})_p$ be the set of tuples $(a_0, a_1, \ldots, a_p, W)$ where

(i) $0 < a_0 < a_1 < \cdots < a_p$ are real numbers,
(ii) $W \in \Psi(N_{(0,\infty)})$, and the a_i are regular values for the projection $p_W\colon W \to [0, \infty)$.

We topologise this as a subspace of $(\mathbb{R}^\delta)^{p+1} \times \Psi(N_{(0,\infty)})$. The collection of all these spaces forms a semi-simplicial space, where the jth face map forgets a_j. It is *not* augmented.

Let $\widehat{D}(N_{(0,\infty)})_p$ be the quotient space of $D(N_{(0,\infty)})_p$ by the relation

$$(a_0, a_1, \ldots, a_p, W) \sim (a'_0, a'_1, \ldots, a'_p, W')$$

if $a_j = a'_j$ for all j and $p_W^{-1}([a_0,\infty)) = p_{W'}^{-1}([a_0,\infty))$. These again form a semi-simplicial space by forgetting the a_j.

There is a semi-simplicial map

$$\pi\colon D(M_{\infty,b};\xi)_\bullet \longrightarrow \widehat{D}(N_{(0,\infty)})_\bullet$$

given by sending a tuple $(a_0, a_1, \ldots, a_p, W)$ to $[a_0, a_1, \ldots, a_p, W \cap N_{(0,\infty)}]$, which factors through the quotient map $r\colon D(N_{(0,\infty)})_\bullet \to \widehat{D}(N_{(0,\infty)})_\bullet$. In addition to these semi-simplicial spaces, we require another pair with stricter requirements.

Definition 8.7. Let $D_\partial(M_{\infty,b};\xi)_\bullet \subset D(M_{\infty,b};\xi)_\bullet$ be the sub-semi-simplicial space where in addition

(i) $W \cap M_{a_0,b}$ is connected,
(ii) each pair $(p_W^{-1}[a_i, a_{i+1}], p_W^{-1}(a_i))$ is connected.

Similarly, let $\widehat{D}_\partial(N_{(0,\infty)})_\bullet \subset \widehat{D}(N_{(0,\infty)})_\bullet$ be the sub-semi-simplicial space where in addition each pair $(p_W^{-1}[a_i, a_{i+1}], p_W^{-1}(a_i))$ is connected. As before, there is a semi-simplicial map $\pi_\partial\colon D_\partial(M_{\infty,b};\xi)_\bullet \to \widehat{D}_\partial(N_{(0,\infty)})_\bullet$ given by restriction.

If $\Sigma \subset L_{[b,c]}$ is a surface satisfying the boundary condition $\xi \subset L_b$ and the boundary condition $\xi' \subset L_c$, we obtain a semi-simplicial map

$$-\cup\Sigma\colon D(M_{\infty,b};\xi)_\bullet \longrightarrow D(M_{\infty,c};\xi')_\bullet$$

over π, and if $(\Sigma, \Sigma\cap L_b)$ is connected then we also obtain a semi-simplicial map

$$-\cup\Sigma\colon D_\partial(M_{\infty,b};\xi)_\bullet \longrightarrow D_\partial(M_{\infty,c};\xi')_\bullet$$

over π_∂.

8.5. Proof of Theorem 1.5 (when $\partial M \neq \emptyset$).

Let us choose once and for all a surface $\Sigma \subset L \times [0,3]$ which satisfies the boundary condition $\xi \subset L$ at both ends (with respect to the obvious collars), is connected, and has positive genus. We define

$$D(M_{\infty,\infty};\xi)_\bullet := \operatorname*{colim}_{b\to\infty} D(M_{\infty,b};\xi)_\bullet,$$

where the colimit is formed using the maps $-\cup\Sigma\colon D(M_{\infty,b};\xi)_\bullet \to D(M_{\infty,b+3};\xi)_\bullet$. We define $D_\partial(M_{\infty,\infty};\xi)_\bullet$ in the same way. Similarly, we define

$$\Psi(M_{\infty,\infty};\xi) := \operatorname*{colim}_{b\to\infty} \Psi(M_{\infty,b};\xi)$$

where the maps in the colimit are again given by union with Σ.

There is a commutative diagram

(8.3)
$$\begin{tikzcd}
D_\partial(M_{\infty,\infty};\xi)_\bullet \arrow[r] \arrow[d, "\pi_\partial"] & D(M_{\infty,\infty};\xi)_\bullet \arrow[r, equal] \arrow[d, "\pi"] & D(M_{\infty,\infty};\xi)_\bullet \arrow[r, "\epsilon_\bullet"] \arrow[d] & \Psi(M_{\infty,\infty};\xi) \arrow[d] \\
\widehat{D}_\partial(N_{(0,\infty)})_\bullet \arrow[r] & \widehat{D}(N_{(0,\infty)})_\bullet & D(N_{(0,\infty)})_\bullet \arrow[l, "r_\bullet"'] \arrow[r, "\epsilon_\bullet"] & \Psi(N_{(0,\infty)})
\end{tikzcd}$$

which we will use to compare the leftmost and rightmost vertical maps after geometric realisation. The first step in doing so is the following.

Lemma 8.8. *The map $r_\bullet$ is a levelwise weak homotopy equivalence, and the two augmentation maps labelled $\epsilon_\bullet$ are weak homotopy equivalences after geometric realisation.*

Proof. The map $r_\bullet$ can be treated with the techniques of [GRW10, Theorem 3.9], and the two augmentation maps can be treated with the techniques of [GRW10, Theorem 3.10]. □

The second step in comparing the leftmost and rightmost vertical maps of (8.3) is to show that the unlabelled horizontal maps are weak homotopy equivalences after geometric realisation. This is much more complicated, and is deferred to Section 9, but we state the result here.

Proposition 8.9. *The maps*

$$|D_\partial(M_{\infty,\infty};\xi)_\bullet| \longrightarrow |D(M_{\infty,\infty};\xi)_\bullet| \quad \text{and} \quad |\widehat{D}_\partial(N_{(0,\infty)})_\bullet| \longrightarrow |\widehat{D}(N_{(0,\infty)})_\bullet|$$

are weak homotopy equivalences.

Before moving on to the proof of this proposition, let us show how we will apply it. We choose a Riemannian metric $\mathfrak{g}$ on $M_{\infty,\infty}$, an $a_0 \in (0,\infty)$, and a function $a\colon M_{\infty,\infty} \to (0,\infty)$ bounded above by the injectivity radius, and so that $\exp_a(TM_{\infty,\infty}|_{N_{[a_0,\infty)}}) \subset N_{(0,\infty)}$. The non-affine scanning map gives the following commutative diagram

$$\begin{tikzcd} \Psi(M_{\infty,b};\xi) \arrow[r, "s_a"] \arrow[d] & \Gamma(\Psi(TM_{\infty,b}) \to M_{\infty,b}; s_a(\xi)) \arrow[d, "\Pi_{a_0,b}"] \\ \Psi(N_{(0,\infty)}) \arrow[r, "s_a"] & \Gamma(\Psi(TN_{[a_0,\infty)}) \to N_{[a_0,\infty)}) \end{tikzcd} \tag{8.4}$$

where both the vertical maps are given by restriction. By Proposition 8.2 the two non-affine scanning maps are weak homotopy equivalences. (For the lower one, we must use that the restriction map

$$\rho\colon \Gamma(\Psi(TN_{(0,\infty)}) \to N_{(0,\infty)}) \longrightarrow \Gamma(\Psi(TN_{[a_0,\infty)}) \to N_{[a_0,\infty)})$$

is an equivalence, and that if we choose a different function a' bounded above by the injectivity radius of $\mathfrak{g}|_{N_{(0,\infty)}}$, then the functions s_a and $\rho \circ s_{a'}$ are homotopic.)

Finally, as $N_{[a_0,\infty)} \hookrightarrow M_{\infty,b}$ is a cofibration, the rightmost vertical map is a fibration, so its homotopy fibre over a section f is equivalent to

$$\Gamma(\Psi(TM_{a_0,b}) \to M_{0,b}; f|_{N_{a_0}}, s_a(\xi)).$$

The following lets us understand the homotopy fibre of $|\pi_\partial|$.

Proposition 8.10. *If $x = (a_0, W) \in \widehat{D}_\partial(N_{(0,\infty)})_0$ then the fibre of $|\pi_\partial|$ over x is*

$$F(a_0,W) := \operatorname*{colim}_{b\to\infty} \left(\coprod_{g\geq 0} \mathcal{E}_{g,c}(M_{a_0,b}; p_W^{-1}(a_0), \xi) \right),$$

where the colimit is formed by $-\cup\Sigma$, and c denotes the number of components of $p_W^{-1}(a_0)\cup\xi$. Furthermore, the map $|\pi_\partial|$ is a homology fibration.

Proof. Identifying the fibre is elementary. To show that $|\pi_\partial|$ is a homology fibration we wish to apply [MS76, Proposition 4]. To do this, we observe that $(\pi_\partial)_p$ is a fibration, and that its fibre over $[a_0, a_1, \ldots, a_p, W]$ is $F(a_0,W)$. Thus face maps d_i for $i > 0$ induce homeomorphisms on fibres, but the face map d_0 induces the map

$$\operatorname*{colim}_{b\to\infty} \left(\coprod_{g\geq 0} \mathcal{E}_{g,c}(M_{a_0,b}; p_W^{-1}(a_0), \xi) \right) \longrightarrow \operatorname*{colim}_{b\to\infty} \left(\coprod_{g\geq 0} \mathcal{E}_{g,c'}(M_{a_1,b}; p_W^{-1}(a_1), \xi) \right)$$

on fibres, given by union with the cobordism $p_W^{-1}([a_0, a_1])$. As this cobordism is connected relative to $p_W^{-1}(a_0)$, union with it may be expressed as a composition of maps of type α, β and γ, so by Theorem 1.3 the induced map on homology is an isomorphism. $\square$

In all, taking geometric realisation and the colimit of diagrams (8.3) and (8.4) over stabilisation of the top row by $- \cup \Sigma$, we obtain a diagram where all horizontal maps are homotopy equivalences. A choice of point $(a_0, W) \in D(N_{(0,\infty)})_0$ such that $p_W^{-1}(a_0) = \emptyset$ gives a compatible collection of basepoints in all the spaces on the bottom row, and we obtain a zig-zag of weak homotopy equivalences between the homotopy fibre of all the vertical maps, taken at this compatible collection of basepoints. In particular, we obtain a zig-zag of homology equivalences between the actual fibres of of $|\pi_\partial|$ and Π_∞,

$$\operatorname*{colim}_{b\to\infty}\left(\coprod_{g\geq 0}\mathcal{E}_{g,c}(M_{a_0,b};\emptyset,\xi)\right) \tag{8.5}$$

and

$$\operatorname*{colim}_{b\to\infty}\left(\Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi))\right). \tag{8.6}$$

Lemma 8.11. *The stabilisation maps between the spaces of sections*

$$\Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi))$$

are homotopy equivalences.

Proof. The stabilisation map is given by union with the section

$$s_a(\Sigma) \in \Gamma_c(\Psi(T(L \times [0,1])) \to L \times [0,1]; s_a(\xi), s_a(\xi)) =: X$$

obtained by scanning the surface Σ. The space X is a homotopy associative H-space, by concatenating intervals and reparametrising. As L was chosen to be diffeomorphic to $\mathbb{R}^{d-1}$, we may choose such a diffeomorphism; this identifies X with

$$\mathrm{map}_c(\mathbb{R}^{d-1} \times [0,1], \Psi(\mathbb{R}^d); s_a(\xi), s_a(\xi)) \simeq \Omega_{s_a(\xi)}(\Omega^{d-1}\Psi(\mathbb{R}^d))$$

as an H-space: in particular $\pi_0(X)$ is a group. Thus there is a section f such that $s_a(\Sigma) \cdot f$ is homotopic to the constant section $s_a(\xi) \times [0,1]$, but then union with the section f gives a homotopy inverse to the stabilisation map. $\square$

Corollary 8.12. *There is a bijection*

$$\pi_0(\Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi))) \cong \mathbb{Z} \times H_2(M;\mathbb{Z}).$$

Proof. The set of path components of (8.5) is $\mathbb{Z} \times H_2(M;\mathbb{Z})$, by Lemma 4.4 . $\square$

In Section 10.1 we will give a concrete description of this bijection. Combining the homology equivalence between (8.5) and (8.6), Lemma 8.11, and Theorem 1.3, we see that the scanning map

$$\mathcal{E}_{g,c}(M_{a_0,b};\emptyset,\xi) \longrightarrow \Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi))$$

is a homology isomorphism in degrees $* \leq \frac{2g-2}{2}$. (We have used the fact that L is contractible, so when we write $\Sigma \subset L \times [0,3]$ as the composition of α and β maps, the β maps are always gluing on a pair of pants with nullhomotopic outgoing boundary, so Theorem 1.3 (ii) gives stability range $* \leq \frac{2g}{3}$ for gluing β maps.)

Extending surfaces and sections cylindrically from M to $M_{a_0,b}$ gives a commutative square

$$\begin{tikzcd} \mathcal{E}_{g,c}(M;\xi) \arrow[r] \arrow[d] & \Gamma(\Psi(TM) \to M; s_a(\xi)) \arrow[d] \\ \mathcal{E}_{g,c}(M_{a_0,b};\emptyset,\xi) \arrow[r] & \Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi)) \end{tikzcd}$$

where the vertical maps are clearly homotopy equivalences; this proves the first part of Theorem 1.5. The second part of Theorem 1.5, *in the case where the manifold M has non-empty boundary*, follows from the commutative square

$$\begin{tikzcd} \mathcal{E}_{g,1}(M;\xi) \arrow[r] \arrow[d, "\gamma_{g,1}"] & \Gamma(\Psi(TM) \to M; s_a(\xi)) \arrow[d] \\ \mathcal{E}_{g}(M_1) \arrow[r] & \Gamma(\Psi(TM_1) \to M_1; s_a(\emptyset)) \end{tikzcd}$$

where $\xi \subset \partial M$ is a single nullhomotopic circle, $\gamma_{g,1}$ is the map that glues on a collar $[0,1] \times \partial M$ containing a disc, and the right-hand map is given by union with the section obtained by scanning the disc. The right-hand map is an equivalence by an argument analogous to that of Lemma 8.11, and the left-hand map is an isomorphism in homology in degrees $* \leq \frac{2g}{3}$ by Theorem 1.3. This finishes the proof of Theorem 1.5 in the case where the manifold M has non-empty boundary. In Section 10 we will show how to deduce Theorem 1.5 in the case where M has empty boundary.

9. Surgery

In this section we prove Proposition 8.9, following the methods in [GMTW09] and [GRW12]. There is an improvement in the way we deal with the complex of surgery data in the sense that the maps used are always simplicial (cf. [GRW12, Section 6]). We will prove in detail that the map

$$|D_\partial(M_{\infty,\infty};\xi)_\bullet| \longrightarrow |D(M_{\infty,\infty};\xi)_\bullet| \tag{9.1}$$

is a weak homotopy equivalence, then briefly explain the changes in the argument to show that

$$|\widehat{D}_\partial(N_{(0,\infty)})_\bullet| \longrightarrow |\widehat{D}(N_{(0,\infty)})_\bullet| \tag{9.2}$$

is a weak homotopy equivalence. We first introduce two more auxiliary semi-simplicial spaces.

Definition 9.1. We define a semi-simplicial space $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$ whose i-simplices are the tuples $(W, a_0, \ldots, a_i)$ such that

(i) $0 < a_0 < a_1 < \cdots < a_i \in \mathbb{R}$.
(ii) $W \in \Psi(M_{\infty,b};\xi)$.
(iii) Each a_j is either a regular value of $p_W \colon W \cap N_{[0,\infty)} \to [0,\infty)$, or $p_W^{-1}(a_j)$ contains only Morse critical points of index at least 1. We denote $\delta_j = p_W^{-1}(a_j)$.
(iv) For each j, the map $\pi_0(\delta_j) \to \pi_0(W \cap N_{[a_j,a_{j+1})})$ induced by the inclusion is a surjection.
(v) $W \cap (M \cup N_{[0,a_0)} \cup L_{[0,b]})$ is path connected.

Similarly, we define $D^\natural(M_{\infty,b};\xi)_\bullet$ to have i-simplices those tuples $(W, a_0, \ldots, a_i)$ which satisfy just the first three conditions above. In both cases, the simplices are topologised as a subspace of $(\mathbb{R}^\delta)^{i+1} \times \Psi(M_{\infty,b};\xi)$, and the face maps are given by forgetting the a_j.

Lemma 9.2. *The inclusions*

$$|D\partial(M_{\infty,b};\xi)_\bullet| \longrightarrow |D^\natural_\partial(M_{\infty,b};\xi)_\bullet| \quad \textit{and} \quad |D(M_{\infty,b};\xi)_\bullet| \longrightarrow |D^\natural(M_{\infty,b};\xi)_\bullet|$$

are weak homotopy equivalances

Proof. The argument is the same in both cases; to be specific we treat the first. Let $\mathcal{J}_{\bullet,\bullet}$ be the bi-semi-simplicial space whose (i,j)-simplices consist of the tuples $(W, a_0, \ldots, a_i, b_0, \ldots, b_j)$ such that $(W, a_0, \ldots, a_i)$ is an i-simplex in $D\partial(M_{\infty,b};\xi)_\bullet$ and $(W, a_0, \ldots, a_i, b_0, \ldots, b_j)$ is an $(i+j+1)$-simplex in $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$.

The $(p,\bullet)$-face map forgets the value a_p and the $(\bullet,q)$-face map forgets the value b_q. It has an augmentation $\epsilon_{-,\bullet}$ to $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$ given by forgetting all the values $a_0, \ldots, a_i$, and an augmentation $\epsilon_{\bullet,-}$ to $D\partial(M_{\infty,b};\xi)_\bullet$ given by forgetting all the values $b_0, \ldots, b_i$. The triangle

$$\begin{tikzcd} & {|\mathcal{J}_{\bullet,\bullet}|} \arrow[dl, "|\epsilon_{\bullet,-}|"'] \arrow[dr, "|\epsilon_{-,\bullet}|"] & \\ {|D\partial(M_{\infty,b};\xi)_\bullet|} \arrow[rr] & & {|D^\natural_\partial(M_{\infty,b};\xi)_\bullet|} \end{tikzcd}$$

commutes up to homotopy, by construction.

The augmentation maps have local sections. We try to define a section of $\epsilon_{i,-}\colon \mathcal{J}_{i,0} \to D\partial(M_{\infty,b};\xi)_i$ through the point $(W, a_0, \ldots, a_i, b_0)$ on the open neighbourhood U of $(W, a_0, \ldots, a_i)$ consisting of those W' such that $a_0, \ldots, a_i$ are still regular values and b_0 contains only Morse critical points of index at least 1, by the formula $(W', a_0, \ldots, a_i) \mapsto (W', a_0, \ldots, a_i, b_0)$. To see that this defines a section, we must check that $p_{W'}^{-1}([a_j, a_{j+1}))$ and $p_{W'}^{-1}([a_i, b_0))$ all satisfy the connectivity requirement (iv). The first case is immediate: as the a_j remain regular values, $p_{W'}^{-1}([a_j, a_{j+1})) \cong p_W^{-1}([a_j, a_{j+1}))$ and $W \cap (M \cup N_{[0,a_0)} \cup L_{[0,b]}) \cong W' \cap (M \cup N_{[0,a_0)} \cup L_{[0,b]})$. In the second case $p_{W'}^{-1}([a_i, b_0))$ differs from $p_W^{-1}([a_i, b_0))$ by adding 1- or 2-handles, but this does not change the connectivity property with respect to the lower boundary. We show that the augmentation map $\epsilon_{-,j}$ has local sections in a similar (but easier) way.

The fiber $F_\bullet$ of $\epsilon_{\bullet,-}$ over $(W, a_0, \ldots, a_i)$ has p-simplices those tuples of real numbers $(b_0, \ldots, b_p)$ such that $(W, a_0, \ldots, a_i, b_0, \ldots, b_p)$ is a simplex of $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$, i.e. $p_W^{-1}(b_j)$ contains only Morse critical points of index at least 1, and $p_W^{-1}([a_i, b_0))$ and each $p_W^{-1}([b_j, b_{j+1}))$ are connected relative to its lower boundary. These conditions only involve pairs of b_j's, so this is a topological flag complex (whose topology is discrete). Given a finite collection $b_1, \ldots, b_n$ of elements of F_0, we may choose a $a_i < c < \min(b_j)$ such that $[a_i, c]$ consists of regular values of p_W. Then c is also in F_0, and $(c, b_j) \in F_1$ for each b_j. It follows from Criterion 2.22 (and Remark 2.23) that $|\epsilon_{\bullet,-}|$ is a weak homotopy equivalence.

The fiber $F'_\bullet$ of $\epsilon_{-,\bullet}$ over $(W, b_0, \ldots, b_j)$ has p-simplices those tuples of real numbers $(a_0, \ldots, a_p)$ which are regular values of p_W, such that $(W, a_0, \ldots, a_p, b_0, \ldots, b_j)$ is a simplex of $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$, which is again seen to be a topological flag complex (whose topology is discrete). For a finite collection $a_1, \ldots, a_n$ of elements of F'_0,

choose a $\max(a_j) < c < b_0$ such that $[c, b_0)$ consists of regular values of p_W. Then c is also in F'_0, and we claim that each (a_j, c) is a 1-simplex of $F'_\bullet$, i.e. that $p_W^{-1}([a_j, c))$ is path-connected relative to $p_W^{-1}(a_j)$. To see this, first note that $p_W^{-1}([a_j, b_0))$ is path connected relative to $p_W^{-1}(a_j)$ by assumption, so there is a path from any point of $p_W^{-1}([a_j, c))$ to $p_W^{-1}(a_j)$ inside of $p_W^{-1}([a_j, b_0))$, but as $[c, b_0)$ consists of regular values $p_W^{-1}([c, b_0))$ is a cylinder, so this path may be homotoped into $p_W^{-1}([a_j, c))$ relative to its ends. It follows from Criterion 2.22 that $|\epsilon_{-,\bullet}|$ is a weak homotopy equivalence. □

9.1. **Local surgery move.** Let $w = (W, a_0, \dots, a_i)$ be a simplex in $D^\natural(M_{\infty,b}; \xi)_\bullet$. First we will construct a path from this i-simplex to a i-simplex $w' = (W', a_0, \dots, a_i)$ in $D^\natural_\partial(M_{\infty,b}; \xi)_\bullet$. In particular, this will prove that the inclusion $D^\natural_\partial(M_{\infty,b}; \xi)_\bullet \to D^\natural(M_{\infty,b}; \xi)_\bullet$ is levelwise 0-connected. In the last section we will use this path to show that it is in fact a homotopy equivalence after geometric realisation.

Let $R(w) = \{W_1, \dots, W_k\}$ be the set of connected components of $W \cap M_{a_0,b}$, and let W_0 be the connected component that contains ξ. Define

$$P_{a,b}(W) = \{\omega \in \pi_0(p_W^{-1}[a, b)) \mid a \notin p_W(\omega)\}, \qquad P(w) = \bigcup_{k=0}^{i-1} P_{a_k,a_{k+1}}(W).$$

Observe that w *is in* $D^\natural_\partial(M; \xi)_\bullet$ *if and only if* $R(w) \cup P(w) = \emptyset$. We define the following subsets of $\mathbb{R}^3$:

$$T' = (\{0\} \times [-3, 3]) \cup ((0, 5] \times \{0\}) \subset \mathbb{R}^2 \subset \mathbb{R}^3$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid d(T', (x, y, z)) < 1, |x| \leq 3, y < 5\}$$

and let $x_1, x_2 \colon T \to \mathbb{R}$ be the first and second coordinate functions.

Definition 9.3. Let $(W, a_0, \dots, a_i)$ be an i-simplex in $D^\natural(M; \xi)_\bullet$. A *local surgery datum for* w is a pair $Q = (\Lambda, e)$ where Λ is a set, and $e \colon \Lambda \times T \to M_{\infty,b}$ is a closed embedding whose restriction $e_{|\{\lambda\}\times T}$ we denote by e_λ such that:

(i) $e^{-1}(W \cap M_{a_i,b}) = \Lambda \times (T \cap x_2^{-1}(\{-3, 3\}))$,
(ii) $(\mathrm{Id}_\Lambda \times x_1)(e^{-1}(W \cap N_{[a_i,\infty)})) \subset \Lambda \times (4, 5)$.
(iii) for each $\lambda \in \Lambda$, $e_\lambda(x_2^{-1}(-3)) \subset W_0 \cap M_{a_0,b}$,
(iv) for each $\omega \in P(w) \cup R(w)$, there is a $\lambda \in \Lambda$ such that $e_\lambda(x_2^{-1}(3)) \subset \omega$,
(v) $\lim_{x\to 5} e_\lambda(x, y, z) = \infty$ for all $\lambda \in \Lambda$ and all (y, z) such that $\sqrt{y^2 + z^2} < 1$,
(vi) for each $\lambda \in \Lambda$ and for each $j = 0, \dots, i$ there is an $\epsilon > 0$ such that for all $a \in (a_j - \epsilon, a_j + \epsilon)$, either $x_2 e_\lambda^{-1}(N_{a_j}) \in (-2, -1)$ or $x_1 e_\lambda^{-1}(N_{a_j}) \in (2, 3)$.

Proposition 9.4. *A local surgery datum* Q *for an* i*-simplex* w *of* $D^\natural(M; \xi)_\bullet$ *determines a path* $\Phi_Q(t)$ *that starts at* w *and ends in an* i*-simplex of* $D^\natural_\partial(M_{\infty,b}; \xi)_\bullet$.

Proof. Consider the 1-parameter family of diffeomorphisms of

$$Y = T \cup \{(x, y, z) \in \mathbb{R}^3 \mid x \leq 5, ||(y, z)|| < 1\}$$

given by

$$h_t(x, y, z) = \begin{cases} (y, x + (x - 3)te^{2 - \frac{1}{1 - ||(y,z)||^2}}, z) & \text{if } ||(y, z)|| < 1, x \leq 3 \\ (x, y, z) & \text{otherwise.} \end{cases}$$

The properties of this family which we will use, and which we leave to the reader to verify, are the following:

(i) h_0 is the identity,
(ii) if $x \in (4,5)$ and $||(y,z)|| < 1/\sqrt{2}$, then $x_1 h_1^{-1}(x,y,z) \in (3,4)$,
(iii) h_t is the identity on $T \cap x_1^{-1}([-\infty, 3))$,
(iv) h_t extends to $\mathbb{R}^3$ with the identity outside T.

The family h_t induces a 1-parameter familiy of maps

$$H_t \colon \Psi(T) \longrightarrow \Psi(T)$$

given by sending a submanifold W to $h_t(W) \cap T$. From the first property of h_t it follows that H_0 is the identity. In Figures 8b and 8c we give a picture of the action of H_t in the dark disc at the bottom of Figure 8a.

Consider now the path η in $\Psi(T)$ given in Figure 9 that starts with the surface $x_2^{-1}(\{-3,3\})$, which is the disjoint union of two open balls in T. It pushes both balls to infinity, joins the balls there and then pulls them backwards. In Figure 9a a picture at time 0 is given. The three vertical circles represent the balls $x_1^{-1}(2)$, $x_1^{-1}(3)$ and $x_1^{-1}(4)$ and the horizontal circle represents the ball $x_2^{-1}(-1)$. The planes in the figure will be given an interpretation later. In Figures 9b, 9c and 9d the ball is pushed to infinity, and in Figures 9e and 9f the surface returns in the shape of a (non-compact) pair of pants. The main properties of this movement are the following:

(i) $\eta(0) = x_2^{-1}(\{-3,3\})$,
(ii) all the values in $(1,2)$ are regular values or Morse critical values of index 2 for the restriction of x_2 to $\eta(t)$,
(iii) all the values in $(2,3)$ are regular values for the restriction of x_1 to $\eta(t)$ or Morse critical values of Morse index 1 or 2 (the former possibility happens only in the step from 9e to 9f),
(iv) $\eta(t) \cap x_1^{-1}((4,5)) \subset \{(x,y,z) \in T \mid ||(y,z)|| < 1/\sqrt{2}, y \in (4,5)\}$,
(v) in the surface $\eta(1)$, the circles $x_2^{-1}(\{-3,3\}) \cap \eta(1)$ are in the same connected component.

Now, let $V \in \Psi(T)$ be the union of the balls $V_0 = x_2^{-1}(\{-3,3\})$ and some surface $V_1 \subset \{(x,y,z) \in T \mid x \in (4,5)\}$. We define a path $\phi_V \colon I \to \Psi(T)$ as

$$\phi_V(t) = \begin{cases} H_{2t}(V) & \text{if } t \in [0,1/2] \\ H_1(V_1) \cup \eta(2t-1) & \text{if } t \in [1/2,1]. \end{cases}$$

Property (iii) of h_t assures that both paths glue well: $H_1(V) = H_1(V_1) \cup V_0 = H_1(V_1) \cup \eta(0)$. Property (ii) for h_t and property (iv) for η assure that the union $H_1(V_1) \cup \eta(2t-1)$ is a union of disjoint surfaces, hence a surface. Hence the path is well-defined. We will use the following properties of this path:

(i) $\phi_V(0) = V$ by Property (i) of h_t,
(ii) $\phi_V(1) \cap x_2^{-1}(\{-3,3\})$ is connected, by property (v) of η.

If we are given a set V_Λ of surfaces $V_\lambda \subset \Psi(\{\lambda\} \times T)$ indexed by λ, we denote by $\phi_{V_\Lambda}(t)$ the result of performing $\phi_{V_\lambda}(t)$ in each $\lambda \times T$.

Now, let Q be a surgery datum for w, and let us define a path Φ_Q in $D^\natural(M_{\infty,b};\xi)_\bullet$ starting at $w = (W, a_0, \dots, a_i)$ as $\Phi_Q(t) = (W_Q(t), a_0, \dots, a_i)$, where

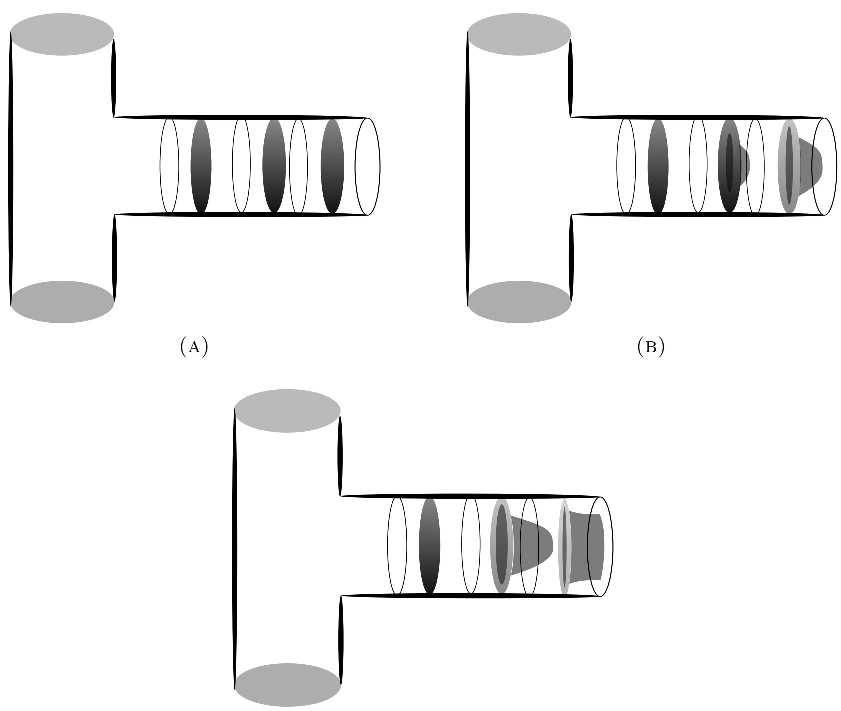
$$W_Q(t) \cap e = e\phi_{e^{-1}(W)}(t), \qquad W_Q(t) \setminus e = W \setminus e.$$

There are five things to check for each $\lambda \in \Lambda$ in order to verify that this path is well-defined. First, that $e_\lambda^{-1}(W)$ is the union of V_0 and some surface V_1 as

above is granted by conditions (i) and (ii) of the surgery data, hence $\phi_{e_\lambda^{-1}(W)}$ is well-defined. Second, that $\Phi_Q(0) = w$ follows from property (i) of ϕ_V. Third, that the union of the two pieces of $W_Q(t)$ is indeed a surface is guaranteed by Property (iv) of h_t. Fourth: as described, the embedding e_λ does not induce a map $\{\lambda\} \times \Psi(T) \to \Psi(M_{\infty,b})$. Condition (v) of the surgery data and properties (iii) and (iv) of h_t grant that the precomposition $I \to \Psi(T) \to \Psi(M_\infty, b)$ with $\phi_{e_\lambda^{-1}(W)}$ is continuous. In other words, they grant that the surface $W_Q(t) \subset M_{\infty,b}$ is closed in $M(\infty)$ and that $W \cap M_{a,b}$ is compact. Fifth, that $(a_0, \dots, a_i)$ are regular values or Morse critical points of index 1 or 2 is a consequence of properties (ii) and (iii) of the path η, together with the following consequence of conditions (v) and (vi) of the surgery data:

(i) If $x_2 e_\lambda^{-1}(a_j) \in (-2, -1)$, then $p^{-1}_{W_Q(t)}(a_j) = x_2^{-1}(b_j)$ for some $b_j \in (1, 2)$
(ii) If $x_1 e_\lambda^{-1}(a_j) \in (2, 3)$, then $p^{-1}_{W_Q(t)}(a_j) = x_1^{-1}(b_j)$ for some $b_j \in (2, 3)$
(iii) $\frac{\partial}{\partial x_1} p_W e_\lambda(x, y, z) > 0$ if $y \in (-2, -1)$
(iv) $\frac{\partial}{\partial x_2} p_W e_\lambda(x, y, z) < 0$ if $x \in (2, 3)$.

Finally, from conditions (ii) and (iii) in the definition of surgery datum and property (ii) of ϕ_V, it follows that $P(\Phi_Q(1)) \cup R(\Phi_Q(1))$ is the empty set, hence $\Phi_Q(1) \in D^\natural(M_{\infty,b}; \xi)_\bullet$. □



(A) (B) (C)

FIGURE 8. The effect of the family H_t in the surgery movement on discs in $x_1^{-1}((2, 3))$, $x_1^{-1}((3, 4))$ and $x_1^{-1}((4, 5))$

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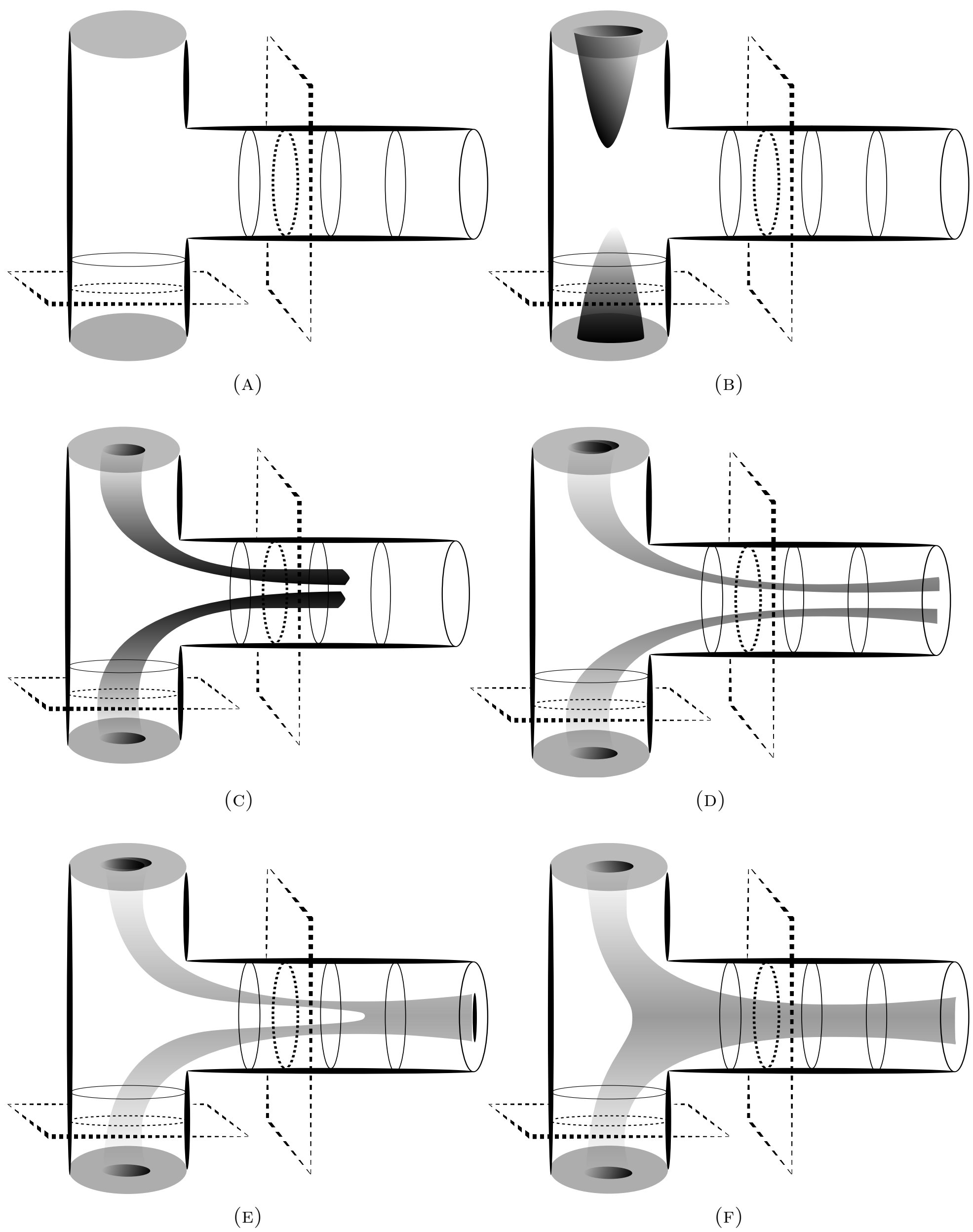

FIGURE 9. The path η in the surgery movement. The shadowed surface at the top of (9a) is $\gamma(t)$, starting with $\gamma(0) = x_2^{-1}(\{-3, 3\})$. The dotted planes are $e^{-1}(N_{a_j})$, and are still planes because of condition (vi) of the local surgery data. The circles drawn correspond to points in $x_1^{-1}(2), x_1^{-1}(3), x_1^{-1}(4)$ and $x_2^{-1}(-2)$.

Remark 9.5. This move is a simplified version of the one used in [GMTW09]. Sadly, that move needs to push parts of the surface to both $+\infty$ and $-\infty$, while here we are only allowed to push things to $+\infty$.

9.2. **Global surgery move.** We will now construct a bi-semi-simplicial space $\mathcal{H}_{\bullet,\bullet}$ augmented over $D^{\natural}(M_{\infty,b};\xi)_\bullet$ which, over each simplex of $D^{\natural}(M_{\infty,b};\xi)_\bullet$, consists of certain tuples of local surgery data. This will allow us to compare it to $D^{\natural}_{\partial}(M_{\infty,b};\xi)_\bullet$ by "doing surgery" in an appropriate way.

Definition 9.6. Let $\mathcal{H}_{\bullet,\bullet}$ be the bisemi-simplicial space whose space of (i,j)-simplices is the space of tuples $(w, Q_0,\dots,Q_j, s_0,\dots,s_j)$ where

(i) w is an i-simplex in $D^{\natural}(M_{\infty,b};\xi)_\bullet$,
(ii) each Q_q is a local surgery datum for w,
(iii) the embeddings in $Q_0,\dots,Q_j$ are pairwise disjoint,
(iv) $(s_0,\dots,s_j)\in[0,1]^{j+1}$.

The $(p,\bullet)$th face map forgets the regular value $a_p\in w$ and the $(\bullet,q)$th face map is

$$\partial_{\bullet,q}(w,Q_0,\dots,Q_i,s_0,\dots,s_i) = (\Phi_{Q_q}(s_q),Q_0,\dots,\hat{Q}_q,\dots,Q_i,s_0,\dots,\hat{s}_q,\dots,s_i).$$

There is an augmentation map $\epsilon_{\bullet,\bullet}$ to $D^{\natural}(M_{\infty,b};\xi)_\bullet$ given by performing the surgery Q_q on w up to time s_q for all q and forgetting all the surgery data. Let $\mathcal{H}^1_{\bullet,\bullet}$ be the bisemi-simplicial subspace of those simplices such that $s_0=\ldots=s_j=1$. Note that by Proposition 9.4 the restriction $\epsilon^1_{\bullet,\bullet}$ of $\epsilon_{\bullet,\bullet}$ to this subspace gives an augmentation onto $D^{\natural}_{\partial}(M_{\infty,b};\xi)_\bullet$ and the following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}^1_{\bullet,\bullet} \arrow[r] \arrow[d, "\epsilon^1_{\bullet,\bullet}"] & \mathcal{H}_{\bullet,\bullet} \arrow[d, "\epsilon_{\bullet,\bullet}"] \\
D^{\natural}_{\partial}(M_{\infty,b};\xi)_\bullet \arrow[r] & D^{\natural}(M_{\infty,b};\xi)_\bullet.
\end{tikzcd} \tag{9.3}$$

Proposition 9.7. *If M has dimension at least 4, the inclusion of $\mathcal{H}^1_{\bullet,\bullet}$ into $\mathcal{H}_{\bullet,\bullet}$ and the augmentation maps are weak homotopy equivalences after geometric realisation.*

The first part of Proposition 8.9 now follows from the commutative diagram

$$\begin{tikzcd}
\lvert D_{\partial}(M_{\infty,b};\xi)_\bullet\rvert \arrow[r] \arrow[d] & \lvert D(M_{\infty,b};\xi)_\bullet\rvert \arrow[d] \\
\lvert D^{\natural}_{\partial}(M_{\infty,b};\xi)_\bullet\rvert \arrow[r] & \lvert D^{\natural}(M_{\infty,b};\xi)_\bullet\rvert,
\end{tikzcd}$$

after taking the limit $b\to\infty$, as the vertical maps are equivalences by Lemma 9.2, and the lower map is an equivalence by (9.3) and Proposition 9.7. As we remarked earlier, the second part of Proposition 8.9 is proved similarly.

Proof of Proposition 9.7. It is clear that the inclusion $\mathcal{H}^1_{\bullet,\bullet}\to\mathcal{H}_{\bullet,\bullet}$ is a levelwise equivalence. To see that the augmentation map $\epsilon^1_{\bullet,\bullet}$ is a homotopy equivalence after geometric realisation, we notice that the augmented semi-simplicial space $\epsilon^1_{i,\bullet}:\mathcal{H}^1_{i,\bullet}\to D^{\natural}_{\partial}(M_{\infty,b};\xi)_i$ has a simplicial contraction, by adding the empty surgery data.

For the map $\epsilon_{\bullet,\bullet}$, let $\mathcal{H}^0_{\bullet,\bullet}$ be the semi-simplicial subspace of $\mathcal{H}_{\bullet,\bullet}$ where the simplices are required to have all s_i equal to 0, and let $\mathcal{H}'_{\bullet,\bullet}$ be the semi-simplicial space defined as $\mathcal{H}^0_{\bullet,\bullet}$, but replacing Condition (iii) in the definition of $\mathcal{H}_{\bullet,\bullet}$ by

(iii′) The restrictions of the embeddings e_q in each $Q_q=(\Lambda_q,e_q)$ to the subspace $\Lambda\times T'\subset\Lambda\times T$ are pairwise disjoint.

Notice that $\mathcal{H}^0_{\bullet,\bullet} \subset \mathcal{H}'_{\bullet,\bullet}$ and the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{H}'_{\bullet,\bullet} \arrow[dr, "\epsilon'_{\bullet,\bullet}"'] & \mathcal{H}^0_{\bullet,\bullet} \arrow[l] \arrow[r] \arrow[d, "\epsilon^0_{\bullet,\bullet}"] & \mathcal{H}_{\bullet,\bullet} \arrow[dl, "\epsilon_{\bullet,\bullet}"] \\
& D^\natural(M_{\infty,b};\xi)_\bullet. &
\end{tikzcd}$$

We will prove that the following statements are true, concluding that the augmentation map $\epsilon_{\bullet,\bullet}$ for $\mathcal{H}_{\bullet,\bullet}$ is a homotopy equivalence after geometric realisation, hence finishing the proof of this proposition.

(i) The inclusion of $\mathcal{H}^0_{\bullet,\bullet}$ into $\mathcal{H}_{\bullet,\bullet}$ is a levelwise homotopy equivalence.
(ii) The inclusion of $\mathcal{H}^0_{\bullet,\bullet}$ into $\mathcal{H}'_{\bullet,\bullet}$ is a levelwise homotopy equivalence.
(iii) The augmentation map $\epsilon'_{\bullet,\bullet}$ is a homotopy equivalence.

Statement (i) is clear. For statement (ii), we will prove that the inclusion $\mathcal{H}^0_{\bullet,\bullet} \to \mathcal{H}'_{\bullet,\bullet}$ is a levelwise weak homotopy equivalence. Consider the deformation

$$h\colon \mathcal{H}'_{i,j} \times (0,1] \to \mathcal{H}'_{i,j}$$

that sends a tuple $(w, Q_0, \dots, Q_i)$ to the tuple $(w, h_t(Q_0), \dots, h_t(Q_i))$, where $h_t(Q_q) = (\Lambda_q, h_t(e_q))$ and $h_t(e_q)(x,y,z) = h_t(tx, y, tz)$. Under this deformation any point eventually ends up, and stays, in the subspace $\mathcal{H}^0_{i,j}$. If $f\colon (D^n, S^{n-1}) \to (\mathcal{H}'_{i,j}, \mathcal{H}^0_{i,j})$ represents a relative homotopy class, then because D^n is compact the map $h(-,t)\circ f$ has image in $\mathcal{H}^0_{i,j}$ for some t, so the homotopy class of f is trivial.

For statement (iii), we notice that $\mathcal{H}'_{i,\bullet} \to D^\natural(M_{\infty,b};\xi)_i$ is an augmented topological flag complex, so we may apply Criterion 2.22 to show that it is a weak homotopy equivalence. Then $\epsilon'_{\bullet,\bullet}$ will be a levelwise equivalence in the i-direction, hence a weak homotopy equivalence after realization.

We will prove in Lemma 9.8 that the augmentation map is surjective and has local sections. Moreover, given $w \in D^\natural_\partial(M_{\infty,b};\xi)_i$ and a non-empty finite collection $(w, Q_0), \dots, (w, Q_j)$ of $(i,0)$-simplices over w, as the dimension of M is > 2, we can perturb the restriction $e_{0|\Lambda\times T'}$ of $e_0 \in Q_0$ to be disjoint from $Q_0, \dots, Q_j$, and any extension e_{j+1} to $\Lambda \times T$ of this perturbation will define a 0-simplex orthogonal to the given ones. □

Lemma 9.8. *The augmentation map $\epsilon'_{i,0}\colon \mathcal{H}'_{i,0} \to D^\natural(M_{\infty,b};\xi)_i$ is surjective and has local sections.*

Proof. First we show that $\epsilon'_{i,0}$ is surjective: if $w \in D^\natural_\partial(M_{\infty,b};\xi)_i$, let $\Lambda = P(w) \cup R(w)$. As M is connected, it is clear that we may take a smooth map $e'\colon \Lambda \times T' \to M_{\infty,b}$ satisfying the restriction of conditions (i), (ii), (iii),(iv) and (v) of the local surgery data to T', except that of being an embedding and that of being disjoint from W outside $e(x_1^{-1}((4,5)))$, $e(0,-3,0)$ and $e(0,3,0)$. As the dimension of M is at least 4, a small perturbation makes it satisfy the latter properties. Again, as the dimension is greater than 3, we may thicken the embedding e' to an embedding $e\colon \Lambda \times T \to M_{\infty,b}$ that satisfies all conditions except (vi), and we may deform e to satisfy this last condition.

Next, we show that $\epsilon'_{i,0}$ has local sections. Let $(w, (\Lambda, e)) \in \mathcal{H}'_{i,0}$. We need to find a neighbourhood U of w in $D^\natural(M_{\infty,b};\xi)_i$ and a section $s\colon U \to \mathcal{H}'_{i,0}$ so that $s(w) = (w, (\Lambda, e))$. Write $w = (W, b_0, \dots, b_i)$, and choose $a > b_i$ a regular value of

p_W. Let U be an open neighbourhood of w in $D^\natural(M_{\infty,b};\xi)_i$ for which a remains regular. The space

$$E := \{((W', b_0, \ldots, b_i), x \in W' \cap M_{a,b}) \in U \times M_{a,b}\}$$

over U is a fibre bundle, and so is locally trivial. Choosing a trivialisation on a smaller neighbourhood U' of w, we obtain a map

$$\psi\colon U' \longrightarrow \mathrm{Emb}(W \cap M_{a,b}, M_{\infty,b}),$$

and using the $\mathrm{Diff}_c(M_{\infty,b})$-locally retractile property of $\mathrm{Emb}(W \cap M_{a,b}, M_{\infty,b})$ we obtain an again smaller neighbourhood U'' and a map

$$\phi\colon U'' \longrightarrow \mathrm{Diff}_c(M_{\infty,b})$$

such that $\phi(W', b_0, \ldots, b_i)(W \cap M_{a,b}) = W' \cap M_{a,b}$ for $(W', b_0, \ldots, b_i) \in U''$.

We now attempt to define a section $s\colon U'' \to \mathcal{H}'_{i,0}$ by

$$s(W', b_0, \ldots, b_i) = ((W', b_0, \ldots, b_i), (\Lambda, \phi(W', b_0, \ldots, b_i) \circ e)).$$

To check that this is indeed a section, we must verify the six properties of Definition 9.3 for this data. Properties (i) and (iii) are immediate from the fact that inside $M_{a,b}$ the data $(W', (\Lambda, \phi(W', b_0, \ldots, b_i) \circ e))$ agrees with the data (W, Q) modified by a diffeomorphism of $M_{\infty,b}$. Property (v) is automatic, and property (ii) holds at the point w and is an open condition, so also hold on some neighbourhood $w \in U''' \subset U''$. Property (vi) holds after perhaps shrinking U'', as then the diffeomorphisms $\phi(U'')$ may be assumed to be supported away from $e \cap p^{-1}(\{b_0, \ldots, b_i\})$.

This leaves Property (iv), which follows from the important observation that if w' is sufficiently close to w, then $P(w')$ and $R(w')$ can only be *smaller* than $P(w)$ and $R(w)$; i.e. the amount of surgery we must do to obtain suitably connected surfaces is upper semi-continuous. More precisely, if w' is sufficiently close to w then $W' \cap (M \cup N_{[0,b_0)} \cup L_{[0,b]})$ is obtained from $W \cap (M \cup N_{[0,b_0)} \cup L_{[0,b]})$ by attaching 1- and 2-handles at b_0, and $p_{W'}^{-1}([b_i, b_{i+1}))$ is obtained from $p_{W'}^{-1}([b_i, b_{i+1}))$ by attaching 1- and 2-handles at b_{i+1}, or subtracting 1- and 2-handles at b_i, neither of which change the required connectivity properties. $\square$

10. Stable homology of the space of surfaces in a closed manifold

In this section we prove Theorem 1.5 for manifolds M with empty boundary. Before doing so, we briefly study the set of path components of the space of sections $\Gamma_c(\mathcal{S}(TM) \to M)$ for such manifolds. We fix a complete Riemannian metric $\mathfrak{g}$ on M.

10.1. **Path components of $\Gamma_c(\mathcal{S}(TM) \to M)$.** The space $\mathcal{S}(TM)$ is a bundle of Thom spaces over M, with fibre over $p \in M$ given by $\mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(T_pM))$, the Thom space of the orthogonal complement to the tautological bundle over the Grassmannian of oriented 2-planes in T_pM. Similarly, we can form the bundle of Grassmannians $q\colon \mathrm{Gr}_2^+(TM) \to M$, which comes equipped with a bundle injection $\gamma_2 \hookrightarrow q^*TM$ from the tautological bundle to the pullback of the tangent bundle of M. We let $\gamma_2^\perp \to \mathrm{Gr}_2^+(TM)$ denote the orthogonal complement to γ_2 in q^*TM.

There is a map

$$c\colon \mathcal{S}(TM) \longrightarrow \mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(TM))$$

given by identifying all the points at infinity. If we choose an orientation of TM there is an induced orientation of $\gamma_2^\perp$, so a Thom class

$$u \in H^{d-2}(\mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(TM)); \mathbb{Z}).$$

There is also an Euler class $e = e(\gamma_2) \in H^2(\mathrm{Gr}_2^+(TM); \mathbb{Z})$, and so a class

$$u \cdot e \in H^d(\mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(TM)); \mathbb{Z}).$$

By abuse of notation, we use the names u and $u \cdot e$ for the cohomology classes on $\mathcal{S}(TM)$ given by $c^*(u)$ and $c^*(u \cdot e)$ respectively.

There are maps

$$\begin{array}{rcccl} \pi\colon \Gamma_c(\mathcal{S}(TM) \to M) & \longrightarrow & H_c^{d-2}(M;\mathbb{Z}) & \longrightarrow & H_2(M;\mathbb{Z}) \\ s & \longmapsto & s^*(u) & \longmapsto & \pi(s) \\ \chi\colon \Gamma_c(\mathcal{S}(TM) \to M) & \longrightarrow & H_c^{d}(M;\mathbb{Z}) & \longrightarrow & H_0(M;\mathbb{Z}) \\ s & \longmapsto & s^*(u \cdot e) & \longmapsto & \chi(s) \end{array}$$

obtained by pulling back the classes e or $u \cdot e$ along a section, and then applying Poincaré duality.

Lemma 10.1. *If M is connected then under the scanning map $\mathscr{S}\colon \mathcal{E}_g(M) \to \Gamma_c(\mathcal{S}(TM) \to M)$ we have*

$$\pi(\mathscr{S}([f\colon \Sigma_g \hookrightarrow M])) = f_*([\Sigma_g]) \in H_2(M;\mathbb{Z})$$
$$\chi(\mathscr{S}([f\colon \Sigma_g \hookrightarrow M])) = 2 - 2g \in \mathbb{Z} = H_0(M;\mathbb{Z}).$$

Proof. The cohomology class $u \in H^{d-2}(\mathcal{S}(TM);\mathbb{Z})$ is Poincaré dual to the class of the submanifold $\mathrm{Gr}_2^+(TM) \subset \mathcal{S}(TM)$, so if s is a (suitably transverse) section then $s^*(u)$ is Poincaré dual to the submanifold $s^{-1}(Gr_2^+(TM))$.

The cohomology class $u \cdot e \in H^d(\mathcal{S}(TM);\mathbb{Z})$ is Poincaré dual to the class of the submanifold $Z \subset \mathrm{Gr}_2^+(TM) \subset \mathcal{S}(TM)$ which is the zero set of a transverse section of $\gamma_2 \to \mathrm{Gr}_2^+(TM)$. Thus if s is a (suitably transverse) section, then $s^*(u \cdot e)$ is Poincaré dual to the set of zeroes of a section of $Ts^{-1}(Gr_2^+(TM))$ which is transverse to the zero section: this is $\chi(s^{-1}(Gr_2^+(TM)))$ by the Poincaré–Hopf theorem.

The map obtained by scanning an embedded submanifold $f(\Sigma_g)$ is suitably transverse, and $s^{-1}(Gr_2^+(TM)) = f(\Sigma_g)$, so the claimed identities hold. □

Proposition 10.2. *If M is connected, so $H_0(M;\mathbb{Z}) = \mathbb{Z}$, then the map χ takes values in $2\mathbb{Z}$. If M is simply connected and of dimension $d \geq 5$, the map*

$$\chi \times \pi\colon \pi_0(\Gamma_c(\mathcal{S}(TM) \to M)) \longrightarrow 2\mathbb{Z} \times H_2(M;\mathbb{Z})$$

is a bijection.

Proof. The space of compactly supported sections is the space of compactly supported lifts along $p\colon \mathcal{S}(TM) \to M$ of the identity map of M. We will use the notation $F = \mathcal{S}(\mathbb{R}^d) = \mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(\mathbb{R}^d))$ for the fibre of the map p, and suppose for simplicity that M is compact.

The map $\mathrm{Gr}_2^+(\mathbb{R}^d) \to \mathrm{Gr}_2^+(\mathbb{R}^\infty)$ induces an isomorphism on cohomology in degrees $* \leq d-1$, so

$$\mathbb{Z}[e(\gamma_2)] \longrightarrow H^*(\mathrm{Gr}_2^+(\mathbb{R}^d);\mathbb{Z})$$

is an isomorphism in degrees $* \leq d-1$, and so

$$u \cdot \mathbb{Z}[e(\gamma_2)] \longrightarrow \widetilde{H}^*(\mathcal{S}(\mathbb{R}^d);\mathbb{Z})$$

is an isomorphism in degrees $* \leq 2d-3$. As there are cohomology classes $u \cdot e^i \in H^*(\mathcal{S}(TM), M; \mathbb{Z})$ restricting to $u \cdot e(\gamma_2)^i$ on the fibre, the bundle p satisfies the conditions of the (relative) Leray–Hirsch theorem in degrees $* \leq 2d-3$, so

$$H^*(M;\mathbb{Z}) \otimes (u \cdot \mathbb{Z}[e(\gamma_2)]) \longrightarrow H^*(\mathcal{S}(TM), M; \mathbb{Z})$$

is an isomorphism in this range of degrees.

Let us show that χ takes even values. As $q^*TM = \gamma_2 \oplus \gamma_2^\perp$, we calculate in the $\mathbb{F}_2$-cohomology of $\mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(TM))$

$$\mathrm{Sq}^2(u) = u \cdot w_2(\gamma_2^\perp) = u \cdot (w_2(\gamma_2) + q^* w_2(M)) = u \cdot e + u \cdot q^* w_2(M)$$

and so pulling back via c we have

$$\mathrm{Sq}^2(u) = u \cdot e + u \cdot p^* w_2(M) \in H^d(\mathcal{S}(TM); \mathbb{F}_2).$$

Thus for any section s we have $\mathrm{Sq}^2(s^*u) = s^*(u \cdot e) + s^*u \cdot w_2(M)$ in the $\mathbb{F}_2$-cohomology of M. However $\mathrm{Sq}^2(s^*u) = v_2(M) \cdot s^*u = w_2(M) \cdot s^*u$ as M is simply connected, and so $s^*(u \cdot e) = 0 \in H^d(M; \mathbb{F}_2)$. Thus $s^*(u \cdot e) \in H^d(M; \mathbb{Z}) = \mathbb{Z}$ is even, as claimed.

We will be required to know $\pi_k(\mathcal{S}(\mathbb{R}^d))$ for $k \leq d$. By considering the cohomology calculation above in degrees $* \leq 2d-3$, we see that as long as $d \geq 4$ then $\mathcal{S}(\mathbb{R}^d)$ has a cell structure whose $(d+1)$-skeleton X consists of a $(d-2)$-cell and a d-cell. Because

$$\mathrm{Sq}^2(u) = u \cdot w_2(\gamma_2^\perp) = u \cdot w_2(\gamma_2) \neq 0$$

we see that the d-cell is attached along a non-trivial map $S^{d-1} \to S^{d-2}$, which must be the Hopf map as long as $d \geq 5$. Thus $X \simeq \Sigma^{d-4}\mathbb{CP}^2$, and it remains to calculate the homotopy groups of this space in degrees $* \leq d$. By the Blakers–Massey theorem, the map of pairs $\pi_*(S^{d-2}, S^{d-1}) \to \pi_*(X, *)$ is an isomorphism for $* \leq 2d-5$, so for $* \leq d$ as we have assumed $d \geq 5$. Calculating using the known stable homotopy groups of spheres in this range shows that

$$\pi_{d-2}(\mathcal{S}(\mathbb{R}^d)) \cong \mathbb{Z} \qquad \pi_{d-1}(\mathcal{S}(\mathbb{R}^d)) = 0 \qquad \pi_d(\mathcal{S}(\mathbb{R}^d)) \cong \mathbb{Z}$$

and also that the Hurewicz map is injective in these degrees. (When $d = 5$ we must use that $\pi_5(S^3) = \mathbb{Z}/2\langle \eta^2 \rangle$, even though it is not in the stable range; this may be found in Toda's book [Tod62].)

Let s_0 and s_1 be two sections of p which have the same value of the invariants π and χ, and let us show that they are fibrewise homotopic. We obtain a diagram

$$\begin{tikzcd}
\{0,1\} \times M \arrow[r, "s_0 \cup s_1"] \arrow[d] & \mathcal{S}(TM) \arrow[r, "p \times u"] \arrow[d, "p"] & M \times K(\mathbb{Z}, d-2) \arrow[d, "proj"] \\
{[0,1] \times M} \arrow[r, "proj"] \arrow[ur, dashed] & M \arrow[r, equal] & M
\end{tikzcd} \tag{10.1}$$

and we must supply the dashed arrow. By obstruction theory, the first possible obstruction lies in

$$H^{d-1}([0,1] \times M, \{0,1\} \times M; \pi_{d-2}(\mathcal{S}(\mathbb{R}^d))) \cong H^{d-2}(M; \mathbb{Z})$$

and must be $\pi(s_0) - \pi(s_1)$, as it agrees with the first possible obstruction for the (trivial) right-hand fibration in (10.1). But we have assumed that $\pi(s_0) - \pi(s_1)$ is zero, so there is no obstruction at this stage. The next possible obstruction lies in

$$H^{d+1}([0,1] \times M, \{0,1\} \times M; \pi_d(\mathcal{S}(\mathbb{R}^d))) \cong H^d(M; \mathbb{Z}),$$

and by comparing it with the trivial bundle $M \times K(\mathbb{Z}, d) \to M$ via $p \times (u \cdot e)$, as above, and using the injectivity of the Hurewicz map, we see that this obstruction vanishes if and only if $\chi(s_0) - \chi(s_1)$ does; we have assumed this. As M has dimension d, there are no higher obstructions to constructing the dotted map, which gives a fibrewise homotopy between the two sections. □

10.2. **Proof of Theorem 1.5 (when $\partial M = \emptyset$).** Recall that we have fixed a complete Riemannian metric $\mathfrak{g}$ on M. Let $\mathcal{E}_g^\nu(M) \subset \mathcal{E}_g^+(M) \times (0, \infty)$ be the space of pairs (W, ϵ), where $W \in \mathcal{E}_g^+(M)$ and ϵ is smaller than the injectivity radius of the exponential map $\exp\colon \nu(W) \to M$; we denote by W^ϵ the image of this embedding. The forgetful map $\mathcal{E}_g^\nu(M) \to \mathcal{E}_g^+(M)$ is a weak homotopy equivalence. We denote by ∞_p the point at infinity of $\mathcal{S}(T_pM)$, and write $\infty = \bigcup_p \infty_p$, and we define the support $\operatorname{supp} f$ of $f \in \Gamma_c(\mathcal{S}(TM) \to M)$ to be the closure of $M \backslash f^{-1}(\infty)$.

Definition 10.3. Let $\mathcal{G}_g(M)_\bullet$ be the semi-simplicial space whose i-simplices are tuples $(W, \epsilon; d_0, \ldots, d_i)$, where

(i) $(W, \epsilon) \in \mathcal{E}_g^\nu(M)$,
(ii) $d_0, \ldots, d_i\colon D^d \hookrightarrow M$ are disjoint embeddings of the closed unit disc into M,
(iii) the geodesic distance from $d_j(0)$ to W is at least ϵ, for all j.

The semi-simplicial structure is as usual given by forgetting data, which gives a semi-simplicial space augmented over $\mathcal{E}_g^\nu(M)$.

Proposition 10.4. *If the dimension of M is at least 3, then $\mathcal{G}_g(M)_\bullet$ is a resolution of $\mathcal{E}_g^\nu(M)$.*

Proof. Let $G_\bullet$ be the semi-simplicial space constructed similarly to the above, with i-simplices consisting of those tuples $(W, \epsilon; d_0, \ldots, d_i)$ such that condition (i) above holds, as well as

(ii′) $d_0, \ldots, d_i\colon D^d \hookrightarrow M$ are embeddings of the closed unit disc into M such that the $d_j(0)$ are distinct,
(iii′) $d_j(0) \cap W = \emptyset$, for all j.

There is an inclusion $\mathcal{G}_g(M)_\bullet \hookrightarrow G_\bullet$, which is a levelwise weak homotopy equivalence, by shrinking the discs and ϵ. Now $G_\bullet$ is an augmented topological flag complex over $\mathcal{E}_g^+(M)$, so we apply Criterion 2.22. The augmentation map is a fibration by Corollary 2.14, hence has local sections, and given any finite (possibly empty) collection $(W, \epsilon, d_0), \ldots, (W, \epsilon, d_i)$ of 0-simplices over (W, ϵ), the complement $M \setminus (W \cup \bigcup d_j(0))$ is a non-empty manifold of dimension at least 3, so there is an embedding d of a closed d-ball into it. Then (W, d) is orthogonal to all the former 0-simplices. □

Proposition 10.5. *There are fibrations*

$$\mathcal{E}_g^\nu(M \setminus \cup d_j(0)) \longrightarrow \mathcal{G}_g(M)_i \longrightarrow C_i(M) =: \mathrm{Emb}(\{0, 1, \ldots, i\} \times D^d, M)$$

where the fiber is taken over the point $(d_0, \ldots, d_i)$.

Proof. This is a consequence of Corollary 2.14. □

In the notation of the last section, we let $\Gamma_c(\mathcal{S}(TM) \to M)_g$ denote the collection of path components $\chi^{-1}(2 - 2g)$. Thus it consists of those sections which have "formal genus g".

Definition 10.6. Let $\mathcal{F}_g(M)_\bullet$ be the semi-simplicial space whose i-simplices are tuples $(f, (d_0, h_0), \ldots, (d_i, h_i))$, where

(i) $f \in \Gamma_c(\mathcal{S}(TM) \to M)_g$,
(ii) $d_0, \ldots, d_i \colon D^d \hookrightarrow M$ are disjoint embeddings of the closed unit disc into M,
(iii) $h_0, \ldots, h_i \colon [0,1] \times M \to \mathcal{S}(TM)$ are homotopies of sections such that

$$h_j(0, -) = f(-), \qquad d_j(0) \notin \operatorname{supp} h_j(1, -),$$

and the homotopy h_j is constant outside of the set $d_j(D^d)$.

The jth face map forgets (d_j, h_j), and forgetting everything but f gives an augmentation to the space $\Gamma_c(\mathcal{S}(TM) \to M)_g$.

Proposition 10.7. *If M has dimension at least 3, then $\mathcal{F}_g(M)_\bullet$ is a resolution of $\Gamma_c(\mathcal{S}(TM) \to M)_g$.*

Proof. Let us define $F_g(M)_\bullet$ as the semi-simplicial space whose i-simplices are tuples $(f, (d_0, h_0), \ldots, (d_i, h_i))$ such that conditions (i) and (iii) above hold and condition (ii) is replaced by

(ii′) $d_0, \ldots, d_i \colon D^d \hookrightarrow M$ are embeddings such that the $d_j(0)$ are distinct,

and whose face maps are given by forgetting data, and it has an augmentation to $\Gamma_c(\mathcal{S}(TM) \to M)_g$ that forgets everything but f. There is an obvious semi-simplicial inclusion $\mathcal{F}_g(M)_\bullet \hookrightarrow F_g(M)_\bullet$ over $\Gamma_c(\mathcal{S}(TM) \to M)_g$, and the lemma will follow from the following statements:

(1) the semi-simplicial inclusion is a levelwise weak homotopy equivalence and
(2) the augmentation of $F_g(M)_\bullet$ is a weak homotopy equivalence.

For the first statement, take a smooth function $\lambda \colon [0, \infty) \to [0,1]$ with $\lambda([1, \infty)) = 0$, $\lambda([0, 1/2]) = 1$. Consider the following deformation $H_s \colon F_g(M)_i \times (0,1] \to F_g(M)_i$ (which restricts to a deformation of $\mathcal{F}_g(M)_i$).

$$H_s(f) = f, \quad H_s(d_j)(y) = d_j(sy), \quad H_s(h_j)(t,x) = \begin{cases} h_j(\lambda(||sy||)t, sy) & \text{if } x = d_j(sy) \\ x & \text{otherwise.} \end{cases}$$

Under this deformation, every i-simplex eventually ends up, and stays, in the subspace $\mathcal{F}_g(M)_i$. If $f \colon (D^n, S^{n-1}) \to (F_g(M)_i, \mathcal{F}_g(M)_i)$ represents a relative homotopy class, then because D^n is compact the map $h(-,t) \circ f$ has image in $\mathcal{F}_g(M)_i$ for some t, so the homotopy class of f is trivial.

For the second statement, note that $F_g(M)_\bullet$ is a topological flag complex augmented over $\mathcal{E}_g^+(M)$ whose augmentation is a fibration by Lemmas 2.16 and 2.8. Given a possibly empty finite collection of 0-simplices $(f, d_0, h_0), \ldots, (f, d_i, h_i)$ over f, we may find an embedding of a disc d_{i+1} such that $d_{i+1}(0)$ is different from the points $d_0(0), \ldots, d_i(0)$. We may also find a homotopy h_{i+1} satisfying condition (iii) for the embedding d_{i+1} and the section f, because the space $\mathcal{S}(T_{d_{i+1}(0)}M)$ is path-connected. Hence the conditions of Criterion 2.22 hold, so the augmentation for $F_g(M)_\bullet$ is a weak homotopy equivalence. $\square$

Proposition 10.8. *There are homotopy fibrations*

$$\Gamma_c(\mathcal{S}(TM \setminus \cup d_j(0)) \to M \setminus \cup d_j(0))_g \longrightarrow \mathcal{F}_g(M)_i \longrightarrow C_i(M),$$

where the fiber is taken over the point $(d_0, \ldots, d_i)$.

Proof. The space $C_i(M)$ is $\mathrm{Diff}_\partial(M)$-locally retractile by Lemma 2.13, and the map is equivariant for the action of $\mathrm{Diff}_\partial(M)$; hence, by Lemma 2.8, this is a locally trivial fibration. The fiber is the space Fib_i of tuples $(f, h_0, \dots, h_i)$ where $f \in \Gamma_c(\mathcal{S}(TM) \to M)_g$ and h_j is a homotopy of f supported in d_j such that $h_j(1, -)$ is a section supported away $d_j(0)$. Since the homotopies h_j have disjoint support, we may compose them. There is a homotopy

$$\begin{aligned} H\colon I \times \mathrm{Fib}_i &\longrightarrow \mathrm{Fib}_i \\ (t, (f, (h_0, \dots, h_i))) &\longmapsto (H_t(f), H_t(h_0), \dots, H_t(h_i)) \end{aligned}$$

where

$$\begin{aligned} H_t(f)(-) &= h_0(t, -) \circ \ldots \circ h_i(t, -) \\ H_t(h_j)(s, -) &= h_j(t + s(1-t), -). \end{aligned}$$

This homotopy deformation retracts Fib_i into the subspace Y of those tuples $(f, h_0, \dots, h_i)$ such that $d_j(0) \notin \operatorname{supp} h_j$ and h_j is the constant homotopy. Finally, there is a map

$$Y \longrightarrow \Gamma_c(\mathcal{S}(TM \setminus \cup d_j(0)) \to M \setminus \cup d_j(0))_g$$

given by sending $(f, h_0, \dots, h_i)$ (recall that these homotopies are all constant) to $f_{|M \setminus \cup d_j(0)}$, and this map is a homeomorphism. $\square$

By condition (iii) of Definition 10.3, the scanning map

$$\mathscr{S}\colon \mathcal{E}^\nu_g(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

constructed using the previously chosen Riemannian metric $\mathfrak{g}$ extends to a semi-simplicial map $\mathscr{S}_\bullet\colon \mathcal{G}_g(M)_\bullet \to \mathcal{F}_g(M)_\bullet$ by sending the tuple $(W, \epsilon, d_0, \dots, d_i)$ to the tuple $(\mathscr{S}(W, \epsilon), (d_0, \mathrm{Id}), \dots, (d_i, \mathrm{Id}))$, where Id denotes the constant homotopy.

Proposition 10.9. *The resolution $\mathscr{S}_\bullet$ of the scanning map is a levelwise homology equivalence in degrees $* \leq \frac{2g-2}{3}$. Hence the scanning map is also a homology equivalence in those degrees.*

Proof. The induced map on space of i-simplices is a map of fibrations over $C_i(M)$, and the map induced on fibres is

$$\mathscr{S}_i\colon \mathcal{E}^\nu_g(M \setminus \cup d_j(0)) \longrightarrow \Gamma_c(\mathcal{S}(TM \setminus \cup d_j(0)) \to M \setminus \cup d_j(0))_g.$$

As $\mathscr{S}_i$ is a scanning map, Theorem 1.5 for surfaces in a manifold with boundary (which was proven in the preceding two sections) asserts that $\mathscr{S}_i$ is a homology equivalence in degrees $* \leq \frac{2g-2}{3}$. Note that although $M \setminus \cup d_j(0)$ does not have boundary, it does admit a boundary. $\square$

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E-mail address: `federico.cantero@ub.edu`

Universitat de Barcelona, Gran via de les corts catalanes, 585, 08007 Barcelona, Spain

E-mail address: `o.randal-williams@dpmms.cam.ac.uk`

DPMMS, Wilberforce Road, Cambridge CB3 0WB, UK